ANALYSIS OF GEOMETRIC OPERATORS ON OPEN MANIFOLDS: A GROUPOID APPROACH

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Abstract. The first five sections of this paper are a survey of algebras of pseudodifferential operators on groupoids. We thus review differentiable groupoids, the definition of pseudodifferential operators on groupoids, and some of their properties. We use then this background material to establish a few new results on these algebras that are useful for the analysis of geometric operators on non-compact manifolds and singular spaces. The first step is to establish that the geometric operators on groupoids are in our algebras. This then leads to criteria for Fredholmness for geometric operators on suitable non-compact manifolds, as well as to an inductive procedure to study their essential spectrum. As an application, we answer a question of Melrose on the essential spectrum of the Laplace operator on manifolds with multi-cylindrical ends.

Contents

Introduction

The first half of this paper is a survey of the results from [13, 29, 30] and [32]. However, there are some new results, including a determination of the essential spectrum of the Laplace operator on complete manifolds with multi-cylindrical ends. This was formulated as a question in [21] (Conjecture 7.1).

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Let us now discuss the contents of this paper. As we mentioned above, the first five sections of this paper are mostly a survey of results on pseudodifferential operators on groupoids. In Section 1, we review some definitions involving manifolds with corners and we introduce groupoids. We also define in this section the class of groupoids we are interested in, namely "differentiable groupoids" (Definition 4), and we recall the definition of the Lie algebroid associated to a differentiable groupoid. (Our "differentiable groupoids" should more properly be called "Lie groupoids." However, this term was already used for some specific classes of differentiable groupoids.) The construction which associates to a differentiable groupoid $\mathcal{G}$ its Lie algebroid $A(\mathcal{G})$ is a generalization of the construction which associates to a Lie group its Lie algebra. This construction is made possible for differentiable groupoids by the fact that the fibers $\mathcal{G}_x := d^{-1}(x)$ of the domain map $d : \mathcal{G} \to M$ consist of smooth manifolds without corners (in this paper, a "smooth manifold" will always mean a "smooth manifold without corners," and a C^∞-manifold with corners will be called simply, a "manifold"). It is convenient to think of $\mathcal{G}$ as a set of arrows between various points, called units, which can be composed according to some definite rules. If $g \in \mathcal{G}$ is such an "arrow," then $d(g)$ is its domain and $r(g)$ is its range, so $\mathcal{G}_x$ is the set of all arrows starting at (or with domain) x.

Section 2 contains several examples of differentiable groupoids. In Section 3, we introduce pseudodifferential operators on groupoids. A pseudodifferential operator P on the differentiable groupoid $\mathcal{G}$ is actually a family $P = (P_x)$ of ordinary pseudodifferential operators P_x on the smooth manifolds without corners $\mathcal{G}_x$. This family is required to be invariant with respect to the natural action of $\mathcal{G}$ by right translations and to be differentiable in a natural sense. Because the family P acts on $\mathcal{C}_{\mathrm{c}}^\infty(\mathcal{G})$, we can regard P as an operator on this space. To get the functoriality and composition properties right, we also assume that the convolution kernel κ_P of P is compactly supported (κ_P has been called "the reduced kernel of P" in [32]). We denote by $\Psi^m(\mathcal{G})$ the space of such order m (families of) pseudodifferential operators on $\mathcal{G}$. Many properties of the usual pseudodifferential operators on smooth manifolds extend to the operators in $\Psi^m(\mathcal{G})$, most notably, we get the existence of the principal symbol map. At the end of the section, we indicate how to treat operators acting between sections of two vector bundles. Section 4 deals with the necessary facts about the actions of $\Psi^\infty(\mathcal{G})$ on various classes of functions. It is convenient to present this from the point of view of representation theory (after all, this is the representation theory of the Lie group G if $\mathcal{G} = G$). It is in fact enough to study representations of $\Psi^{-\infty}(\mathcal{G})$, and this includes the study of boundedness properties of various representations: to check that a representation of $\Psi^0(\mathcal{G})$ consists of bounded operators, it is sufficient to check that its restriction to $\Psi^{-\infty}(\mathcal{G})$ is bounded. This generalizes the usual boundedness theorems pseudodifferential operators of order 0.

Motivated by a question of Connes, Monthubert also was lead to define pseudodifferential operators on groupoids in [26], independently. Actually, Connes had defined algebras of pseudodifferential operators on the graphs of $C^{0,\infty}$ foliations in [2], before. The definitions of pseudodifferential operators on groupoids in [26] and [32] is essentially the same as Connes', but different because we take into account the differentiability in the transverse direction, too, and, most importantly, we allow non-regular Lie algebroids. An approach to operators on singular spaces which is similar in philosophy to ours was outlined in [21]. There Melrose considers operators whose kernels are defined on compact manifolds with corners and have

structural maps that make them "semi-groupoids." The results of [32] were first presented in July 1996 at the joint SIAM-AMS-MAA Meeting on Quantization in Mount Holyoke.

In addition to the survey of the results from [13, 29, 30] and [32], the first five sections of the paper also include many examples of groupoids together with the description of the resulting algebras of pseudodifferential operators. Actually, Sections 2 and 5 are devoted exclusively to examples, with the hope that this will make the general theory easier to apply. Whenever it was relevant, we have also compared our construction to the classical constructions.

Let us now quickly describe the contents of each of the remaining five sections of this paper. In Section 6 we show that the geometric differential operators acting on the fibers of the domain map $d : \mathcal{G} \to M$ belong to our algebras $\Psi^\infty(\mathcal{G}; E)$, for suitable E. To define these geometric operators – except the de Rham operator – we need a metric on $\mathcal{G}_x$, and this will come from a metric on $A(\mathcal{G})$, the Lie algebroid of $\mathcal{G}$. Then we need to establish the existence of right invariant connections with the properties necessary to define the geometric operators we are interested in. It needs to be established, for example, that compatible connections exist on Clifford modules, and this is not obvious in the groupoid case. Section 7 establishes the technical facts needed to define Sobolev spaces in our setting. The reader should find the statements in that section easy to understand and believe (although, unfortunately, not the same thing can be said about their proofs).

Beginning with Section 8, we begin to work with groupoids $\mathcal{G}$ of a special kind, which model operators on suitable non-compact Riemannian manifolds (M_0, g). The set of units M of these groupoids is a compactification of M_0 to a manifold with corners, and hence we can think of $\mathcal{G}$ as modeling the behavior at ∞ of M_0 (this approach was inspired in part by Melrose's geometric scattering theory program). More precisely, we assume that M_0 is an open invariant subset of M with the property that $\mathcal{G}_{M_0}$ is the pair groupoid, that is, that for each $x \in M_0$ there exists exactly one arrow to any other point $y \in M_0$, and there exists no arrow to any point not in M_0 (see Example 3). Then the restriction of $A(\mathcal{G})$ to M_0 identifies canonically with the tangent bundle to M_0. For such Riemannian manifolds, the geometric operators on M_0 can be recovered from the geometric operators on $\mathcal{G}$. We use these results in Section 10 to study the Hodge-Laplace operators on suitable non-compact Riemannian manifolds, using the general spectral properties discussed in Section 9. We obtain, in particular, criteria for certain pseudodifferential operators on M_0 to be Fredholm or compact, similar to the well-known criteria for b-pseudodifferential operators to be Fredholm or compact [22, 24]

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1. Manifolds with corners and groupoids

In the following, we shall consider manifolds with corners. Brief introductions into the analysis on manifolds with corners can be found for instance in [24], and

will be discussed in more detail in a forthcoming book of Melrose. We begin this section with a short discussion of the relevant definitions concerning manifolds with corners. Then we introduce groupoids and the class of *differentiable groupoids*. The reader can find more information on groupoids in [3, 38].

By a *manifold* we shall always understand a differentiable manifold possibly *with corners*, by a *smooth manifold* we shall always mean a differentiable manifold *without corners*. By definition, every point p in a manifold with corners M has a coordinate neighborhood diffeomorphic to $[0,\infty)^k \times \mathbb{R}^{n-k}$ such that the transition functions are smooth (including on the boundary). Moreover, we assume that each boundary hyperface H of M is an embedded submanifold and has a defining function, that is, that there exists a smooth function $x \geq 0$ on M such that

$$H = \{x = 0\} \text{ and } dx \neq 0 \text{ on } H\,.$$

It follows that if $H_1, \dots, H_k$ are boundary hyperfaces of M with defining functions $x_1, \dots, x_k$, then the differentials $dx_1, \dots, dx_k$ are linearly independent at the intersection $H_1 \cap \ldots \cap H_k$.

Definition 1. *A* <u>*submersion*</u> *$f : M \to N$, between two manifolds with corners M and N, is a differentiable map such that df is surjective at all points and $df(v)$ is an inward pointing tangent vector of N if, and only if, v is an inward pointing vector M.*

It is not difficult to prove the following lemma.

Lemma 1. *Let $f : M \to N$ be a submersion of manifolds with corners, $y \in N$ a point belonging to the interior of a face of codimension k and $x_1, x_2, \dots, x_k$ be the defining functions of the hyperfaces containing y. Then $x_1 \circ f, x_2 \circ f, \dots, x_k \circ f$ are defining functions for k distinct hyperfaces of M. Let $p \in M$ be such that $f(p) = y$, then all hyperfaces of M containing p are obtained in this way.*

Proof. Let p and y be as in the statement and $z_j := x_j \circ f$. Because df is surjective, it follows that dz_j are linearly independent at p. Each function z_j is the defining function of a hyperface because f must map faces of codimension k to faces of codimension k. □

For any submersion f as above, it follows that the fibers $f^{-1}(y)$ of f are smooth manifolds (that is *without* corners). We can see this as follows. Because this property of f is local, we can fix y and replace M and N with some small open neighborhoods of y and $f(y)$, respectively. By decreasing these neighborhoods, we can also assume that they are diffeomorphic to one of the model open sets $[0,1)^k \times \mathbb{R}^{n-k}$. Then we extend M and N to smooth open manifolds without corners, denoted $\tilde{M}$ and $\tilde{N}$, and f to a smooth function $\tilde{f} : \tilde{M} \to \tilde{N}$. By decreasing M and N again, if necessary, we can assume that $\tilde{f}$ is still a submersion, this time in the classical sense, because $\tilde{M}$ and $\tilde{N}$ are smooth manifolds. This gives then that $\tilde{f}^{-1}(y)$ is a smooth submanifold of $\tilde{M}$. Since $f^{-1}(y) = \tilde{f}^{-1}(y) \cap M$, it is enough to check that this intersection coincides with a component of $\tilde{f}^{-1}(y)$. Let x be a defining function of N in $\tilde{N}$ (that is, the defining function of a hyperface of N). Then $x \circ f$ is a defining function of some hyperface of M. By counting the hyperfaces of M and N, we see that we get in this way all defining functions of M. Since they all vanish on y, $f^{-1}(y) \cap \{x \circ f \geq 0\} = f^{-1}(y)$. This completes the argument.

The concept of a "submanifold" means the following in our setting.

Definition 2. *A submanifold (or submanifold with corners) N of a manifold with corners M is a submanifold $N \subset M$ such that N is a manifold with corners and each hyperface F of N is a connected component of a set of the form $F' \cap N$, where F' is a hyperface of M intersecting N transversally.*

We shall need groupoids endowed with various structures. ([38] is a general reference for some of what follows.) We recall first that a *small category* is a category whose class of morphisms is a set. The class of objects of a small category is then a set as well. Here is now a quick definition of groupoids.

Definition 3. *A groupoid is a small category $\mathcal{G}$ all of whose morphisms are invertible.*

Let us make this definition more explicit. A groupoid $\mathcal{G}$ is a pair $(\mathcal{G}^{(0)}, \mathcal{G}^{(1)})$ of sets together with structural morphisms d, r, μ, u, and ι. Here the first set, $\mathcal{G}^{(0)}$, represents the objects (or units) of the groupoid and the second set, $\mathcal{G}^{(1)}$, represents the set of morphisms of $\mathcal{G}$. Usually, we shall denote the space of units of $\mathcal{G}$ by M and we shall identify $\mathcal{G}$ with $\mathcal{G}^{(1)}$. Each object of $\mathcal{G}$ can be identified with a morphism of $\mathcal{G}$, the identity morphism of that object, which leads to an injective map $u : M := \mathcal{G}^{(0)} \to \mathcal{G}$, used to identify M with a subset of $\mathcal{G}$. Each morphism $g \in \mathcal{G}$ has a "domain" and a "range." We shall denote by $d(g)$ the *domain* of g and by $r(g)$ the *range* of g. We thus obtain functions

$$d, r : \mathcal{G} \longrightarrow M := \mathcal{G}^{(0)}. \tag{1}$$

The multiplication (or composition) $\mu(g, h) = gh$ of two morphisms $g, h \in \mathcal{G}$ is not always defined; it is defined precisely when $d(g) = r(h)$. The inverse of a morphism g is denoted by $g^{-1} = \iota(g)$.

A groupoid $\mathcal{G}$ is completely determined by the spaces $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ and the structural maps d, r, μ, u, ι. We sometimes write $\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, d, r, \mu, u, \iota)$. The structural maps satisfy the following properties:

- $r(gh) = r(g)$ and $d(gh) = d(h)$, for any pair g, h with $d(g) = r(h)$;
- The partially defined multiplication μ is associative;
- $d(u(x)) = r(u(x)) = x$, $\forall x \in \mathcal{G}^{(0)}$, $u(r(g))g = g$, $gu(d(g)) = g$, $\forall g \in \mathcal{G}^{(1)}$, and $u : \mathcal{G}^{(0)} \to \mathcal{G}^{(1)}$ is one-to-one;
- $r(g^{-1}) = d(g)$, $d(g^{-1}) = r(g)$, $gg^{-1} = u(r(g))$, and $g^{-1}g = u(d(g))$.

Definition 4. *A differentiable groupoid is a groupoid*

$$\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, d, r, \mu, u, \iota)$$

such that $M := \mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ are manifolds with corners, the structural maps d, r, μ, u, and ι are differentiable, the domain map d is a submersion, and all the spaces M and $\mathcal{G}_x := d^{-1}(x)$, $x \in M$, are Hausdorff.

Note that we *do not* require $\mathcal{G}^{(1)}$ to be Hausdorff. We actually need this generality to treat algebras associated to foliations and other geometric structures in our setting.

We also observe that ι is a diffeomorphism, and hence d is a submersion if, and only if, $r = d \circ \iota$ is a submersion. The reason for requiring d to be a submersion is that then each fiber $\mathcal{G}_x = d^{-1}(x) \subset \mathcal{G}^{(1)}$ is a smooth manifold without corners.

We now recall the definition of the "Lie algebroid" of a differentiable groupoid. Lie algebroids are for differentiable groupoids what Lie algebras are for Lie groups.

Definition 5. *A* <u>*Lie algebroid*</u> *A over a manifold M is a vector bundle A over M, together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A and a bundle map $\varrho : A \to TM$, extended to a map between sections of these bundles, such that*

(i) $\varrho([X,Y]) = [\varrho(X), \varrho(Y)]$, and

(ii) $[X, fY] = f[X,Y] + (\varrho(X)f)Y$,

for any smooth sections X and Y of A and any smooth function f on M. The map ϱ is called the anchor.

Note that we allow the base M in the definition above to be a manifold with corners.

The Lie algebroid associated to a differentiable groupoid $\mathcal{G}$ is defined as follows [17, 35]. The vertical tangent bundle (along the fibers of d) of a differentiable groupoid $\mathcal{G}$ is, as usual,

$$T_{vert}\mathcal{G} = \ker d_* = \bigcup_{x \in M} T\mathcal{G}_x \subset T\mathcal{G}. \tag{2}$$

Then $A(\mathcal{G}) := T_{vert}\mathcal{G}\big|_M$, the restriction of the d-vertical tangent bundle to the set of units, M, determines $A(\mathcal{G})$ as a vector bundle.

We now construct the bracket defining the Lie algebra structure on $A(\mathcal{G})$. The right translation by an arrow $g \in \mathcal{G}$ defines a diffeomorphism

$$R_g : \mathcal{G}_{r(g)} \ni g' \longmapsto g'g \in \mathcal{G}_{d(g)}.$$

A vector field X on $\mathcal{G}$ is called d-vertical if $d_*(X(g)) = 0$ for all g. The d-vertical vector fields are precisely the vector fields on $\mathcal{G}$ that can be restricted to the submanifolds $\mathcal{G}_x$. It makes sense then to consider right–invariant vector fields on $\mathcal{G}$. It is not difficult to see that the sections of $A(\mathcal{G})$ are in one-to-one correspondence with d–vertical, right–invariant vector fields on $\mathcal{G}$.

The Lie bracket $[X,Y]$ of two d–vertical right–invariant vector fields X and Y is also d–vertical and right–invariant, and hence the Lie bracket induces a Lie algebra structure on the sections of $A(\mathcal{G})$. To define the action of the sections of $A(\mathcal{G})$ on functions on M, observe that the right invariance property makes sense also for functions on $\mathcal{G}$, and that $\mathcal{C}^\infty(M)$ may be identified with the subspace of smooth right–invariant functions on $\mathcal{G}$, because r is a submersion. If X is a right–invariant vector field on $\mathcal{G}$ and f is a right–invariant function on $\mathcal{G}$, then $X(f)$ will still be a right invariant function. This identifies the action of $\Gamma(A(\mathcal{G}))$ on $\mathcal{C}^\infty(M)$.

We denote by $T^*_{vert}\mathcal{G}$ the dual of $T_{vert}\mathcal{G}$, and by $A^*(\mathcal{G})$ the dual of $A(\mathcal{G})$. Later on, we shall need the bundle Ω^λ_d of λ-densities along the fibers of d. It is defined as follows. If the fibers of d have dimension n, then

$$\Omega^\lambda_d := |\Lambda^n T^*_{vert}\mathcal{G}|^\lambda = \cup_x \Omega^\lambda(\mathcal{G}_x).$$

By invariance, these bundles can be obtained as pull-backs of bundles on M. For example $T_{vert}\mathcal{G} = r^*(A(\mathcal{G}))$ and $\Omega^\lambda_d = r^*(\mathcal{D}^\lambda)$, where $\mathcal{D}^\lambda$ denotes $\Omega^\lambda_d|_M$. If E is a (smooth complex) vector bundle on the set of units M of $\mathcal{G}$, then the pull-back bundle $r^*(E)$ on $\mathcal{G}$ will have right invariant connections obtained as follows. A

connection ∇ on E lifts to a connection on $r^*(E)$. Its restriction to any fiber $\mathcal{G}_x$ defines a linear connection in the usual sense, which is denoted by ∇_x. It is easy to see that these connections are right invariant in the sense that

$$R_g^*\nabla_x = \nabla_y, \quad \forall g \in \mathcal{G} \text{ such that } r(g) = x \text{ and } d(g) = y. \tag{3}$$

The bundles considered above will thus have invariant connections.

We observe that in all considerations above, we first use the smooth structure on each $\mathcal{G}_x$ to define the geometric quantities we are interested in: $X(f)$, $[X, Y]$, and so on. We do need then, however, to check that these quantities define global objects on the possibly non-Hausdorff manifold $\mathcal{G}$, more precisely, we need the defined objects to be smooth on $\mathcal{G}$, not just on every $\mathcal{G}_x$. All these global smoothness conditions can be checked on *smooth* functions on $\mathcal{G}$, as long as we correctly define this concept. For non-Hausdorff manifolds, the correct choice is the one considered by Crainic and Moerdijk in [4], more precisely, we consider first the spaces $V = \oplus_\alpha \mathcal{C}_c^\infty(U_\alpha)$ and $W = \oplus_{\alpha,\beta}\mathcal{C}_c^\infty(U_\alpha \cap U_\beta)$, where $U_\alpha \subset \mathcal{G}$ are the domains of coordinate charts. Then there is a natural map $\delta : W \to V$, which is the direct sum of the maps $(j, -j) : \mathcal{C}_c^\infty(U_\alpha \cap U_\beta) \to \mathcal{C}_c^\infty(U_\alpha) \oplus \mathcal{C}_c^\infty(U_\beta)$, with j the natural inclusion, and we define

$$\mathcal{C}_c^\infty(\mathcal{G}) = V/\delta(W). \tag{4}$$

A function f on $\mathcal{G}$ is *smooth* if, and only if, $\phi f \in \mathcal{C}_c^\infty(U_\alpha)$ for all $\phi \in \mathcal{C}_c^\infty(U_\alpha)$.

If $A \to M$ is a given Lie algebroid and $\mathcal{G}$ is a differentiable groupoid whose Lie algebroid is isomorphic to A, then we say that $\mathcal{G}$ *integrates* A. Not every Lie algebroid can be integrated (see [1] for an example). There are many simple-minded Lie algebroids for which the standard integration procedures of [29] lead to non-Hausdorff groupoids. For many Lie algebroids (including the tangent bundles to foliations) there are no Hausdorff groupoids integrating them. The integration of Lie algebroids was also studied by Moerdijk in [25].

2. Examples I: Groupoids

We now discuss examples of differentiable groupoids. The examples included in this section are either basic examples or theoretical examples predicted by the general theory of groupoids. The examples that we shall use to study geometric operators on open manifolds will be included in Section 5. For each example considered in this section we also describe the corresponding Lie algebroid.

As in the previous section, we shall identify $\mathcal{G}$ with its set of arrows: $\mathcal{G} = \mathcal{G}^{(1)}$, and we shall denote by M or by $\mathcal{G}^{(0)}$ the set of units of $\mathcal{G}$.

Example 1. *Manifolds with corners:* If M is a manifold with corners, then M is naturally a differentiable groupoid with no arrows other than the units, *i.e.,* we have $\mathcal{G} = \mathcal{G}^{(1)} = \mathcal{G}^{(0)} = M$, $d = r = id$.

Then $A(\mathcal{G}) = 0$, the zero bundle on M.

Example 2. *Lie groups:* Every differentiable groupoid $\mathcal{G}$ with space of units consisting of just one point, $M = \{e\}$, is necessarily a Lie group. Conversely, every Lie group G can be regarded as an differentiable groupoid $\mathcal{G} = G$ with exactly one unit $M = \{e\}$, the unit of that group.

In this case $A(\mathcal{G}) = Lie(G)$, the Lie algebra of G.

The above two sets of examples are in a certain way the two extreme examples of differentiable groupoids. The first set of examples consists of groupoids each of which has the least set of arrows among all groupoids with the same set of units. The second set of examples consists of the differentiable groupoids with the least set of units among all non-empty groupoids. The fact that manifolds and Lie groups are particular cases of groupoids makes them a favorite object of study in non-commutative geometry.

The following example plays a special role in the theory pseudodifferential operators on groupoids.

Example 3. *The pair groupoid:* Let M be a smooth manifold (without corners) and let

$$\mathcal{G} = M \times M \quad \text{and} \quad \mathcal{G}^{(0)} = M,$$

with structural morphisms $d(x, y) = y$, $r(x, y) = x$, $(x, y)(y, z) = (x, z)$, $u(x) = (x, x)$ and $\iota(x, y) = (y, x)$. Then $\mathcal{G}$ is a differentiable groupoid, called *the pair groupoid*.

Denote the pair groupoid with units M by $M \times M$. Then $A(M \times M) = TM$, the tangent bundle to M.

A variant of the above example is the following.

Example 4. *The fibered pair groupoid:* Let $p : Y \to B$ be a submersion of manifolds with corners (see Definition 1 of the previous section). The fibered pair groupoid $\mathcal{G}$ is obtained as

$$\mathcal{G} = Y \times_B Y := \{(x, y), p(x) = p(y), x, y \in Y\},$$

with the operations induced from the pair groupoid $Y \times Y$. Its space of units is Y.

The Lie algebroid of $Y \times_B Y$ is $A(\mathcal{G}) = T_{vert}Y$, the kernel of $TY \to TB$, or, in other words, the vertical tangent bundle to the submersion $p : Y \to B$.

Example 5. *Products:* The product $\mathcal{G}_1 \times \mathcal{G}_2$ of two differentiable groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ is again a differentiable groupoid for the product structural morphisms.

To obtain the Lie algebroid of the product groupoid we use the external product of vector bundles (not the direct sum): $A(\mathcal{G}_1 \times \mathcal{G}_2) = A(\mathcal{G}_1) \times A(\mathcal{G}_2)$, a vector bundle over $\mathcal{G}_1^{(0)} \times \mathcal{G}_2^{(0)}$.

We now include some examples that are suggested by the general theory of differentiable groupoids.

Example 6. *Bundles of Lie groups:* In this example $\mathcal{G}$ is a fiber bundle $p : \mathcal{G} \to B$ such that each fiber $\mathcal{G}_b := p^{-1}(b)$ has a Lie group structure, and the induced map

$$\mathcal{G} \times_B \mathcal{G} := \{(g, g') \in \mathcal{G} \times \mathcal{G}, p(g) = p(g')\} \ni (g, g') \mapsto g^{-1}g' \in \mathcal{G}$$

is a smooth map. We define $d = r = p$. The units of the groups $\mathcal{G}_b$ then form a submanifold of $\mathcal{G}$ diffeomorphic to B via p. We do not assume that the fibers $\mathcal{G}_b$ are all isomorphic, but this is true in most cases of interest.

Let $\mathfrak{g}$ be the restriction to the space of units of the vertical tangent bundle to the fibration $\mathcal{G} \to B$. Then $\mathfrak{g}$ is a vector bundle over B and $A(\mathcal{G}) = \mathfrak{g}$. The fiber of $\mathfrak{g}$ above b is then the Lie algebra of $\mathcal{G}_b$, and $\mathfrak{g}$ is a bundle of Lie algebras. In this example, the anchor map $\varrho : A(\mathcal{G}) \to TB$ is the zero map.

Example 7. *Fibered products:* Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two differentiable groupoids with units M_1 and M_2. We assume that both M_1 and M_2 come equipped with submersions $p_i : M_i \to B$, $i = 1, 2$, for some common manifold with corners B. Suppose that for each arrow $g \in M_i$, $p_i(d(g)) = p_i(r(g)) =: p_i(g)$. The fibered product of $\mathcal{G}_1$ with $\mathcal{G}_2$ (with respect to p_i) is then

$$\mathcal{G}_1 \times_B \mathcal{G}_2 := \{(g_1, g_2) \in \mathcal{G}_1 \times_B \mathcal{G}_2, p_1(g_1) = p_2(g_2)\},$$

with product (and structural maps, in general) induced from the product groupoid $\mathcal{G}_1 \times \mathcal{G}_2$.

We get $A(\mathcal{G}_1 \times_B \mathcal{G}_2) = A(\mathcal{G}_1) \times_B A(\mathcal{G}_2)$.

We are particularly interested in the above example when $\mathcal{G}_1 = Y \times_B Y$ is the fibered pair groupoid of Example 4 and $\mathcal{G}_2$ is a bundle of Lie groups with base B. This situation seems to be fundamental in the study of pseudodifferential operators associated to various Lie algebras of vector fields. It will be used for example in Example 22. An index theorem in this framework was obtained in [30], if the fibers of $\mathcal{G}_2 \to B$ are simply-connected solvable.

We continue with some more elaborate examples.

Example 8. *The graph of a foliation:* This groupoid was introduced in [48]. Let (M, F) be a foliated manifold. Thus $F \subset TM$ is an integrable bundle. The graph of (M, F) consists of equivalence classes $[\gamma]$ of paths γ which are completely contained in a leaf, with respect to the equivalence relation $[\gamma] = [\gamma']$ if, and only if, γ and γ' have the same holonomy (this implies, in particular, that they have the same end-points).

The Lie algebroid is $A(\mathcal{G}) = F$ in this example.

Example 9. *The fundamental groupoid:* Let $\mathcal{G}$ be the fundamental groupoid of a compact smooth manifold M (without corners) with fundamental group $\pi_1(M) = \Gamma$. Recall that if we denote by $\widetilde{M}$ a universal covering of M and let Γ act by covering transformations on $\widetilde{M}$, then we have $\mathcal{G}^{(0)} = \widetilde{M}/\Gamma = M$, $\mathcal{G} = (\widetilde{M} \times \widetilde{M})/\Gamma$, and d and r are the two projections. Each fiber $\mathcal{G}_x$ can be identified with $\widetilde{M}$, uniquely up to the action of an element in Γ.

The Lie algebroid is TM, as in the first example.

The following example generalizes the tangent groupoid of Connes; here we closely follow [3, II,5]. A similar construction was used in [6] to define the so called "normal groupoid," which is, anticipating a little bit, the adiabatic groupoid of a foliation.

Example 10. *The adiabatic groupoid:* The adiabatic groupoid $\mathcal{G}_{\mathrm{ad}}$ associated to a differentiable groupoid $\mathcal{G}$ is defined as follows. The space of units is

$$\mathcal{G}_{\mathrm{ad}}^{(0)} := [0, \delta) \times \mathcal{G}^{(0)}\,, \quad \delta > 0,$$

with the product manifold structure. The set of arrows is defined as the disjoint union

$$\mathcal{G}_{\mathrm{ad}} := A(\mathcal{G}) \cup (0, \delta) \times \mathcal{G}.$$

The structural maps are defined as follows. The domain and range are:

$$d(t, g) = (t, d(g)) \quad r(t, g) = (t, r(g))\,, \quad t > 0,$$

and $d(v) = r(v) = (0,x)$, if $v \in T_x\mathcal{G}_x$. The composition is $\mu(\gamma,\gamma') = (t, gg')$, if $t > 0$, $\gamma = (t,g)$, and $\gamma' = (t,g')$, and

$$\mu(v,v') = v + v' \quad \text{ if } v, v' \in T_x\mathcal{G}_x.$$

The smooth structure on the set of arrows is the product structure for $t > 0$. In order to define a coordinate chart at a point

$$v \in T_x\mathcal{G}_x = A_x(\mathcal{G}) = d^{-1}(0,x),$$

choose first a coordinate system $\psi : U = U_1 \times U_2 \to \mathcal{G}$, $U_1 \subset \mathbb{R}^p$ and $U_2 \subset \mathbb{R}^n$ being open sets containing the origin, U_2 convex, with the following properties: $\psi(0,0) = x \in M \subset \mathcal{G}$, $\psi(U) \cap M = \psi(U_1 \times \{0\})$, and there exists a diffeomorphism $\phi : U_1 \to \mathcal{G}^{(0)}$ such that $d(\psi(s,y)) = \phi(s)$ for all $y \in U_2$ and $s \in U_1$.

We identify, using the differential $\mathrm{D}_2\psi$ of the map ψ, the vector space $\{s\} \times \mathbb{R}^n$ and the tangent space $T_{\phi(s)}\mathcal{G}_{\phi(s)} = A_{\phi(s)}(\mathcal{G})$. We obtain then coordinate charts $\psi_\varepsilon : A(\mathcal{G})|_{\phi(U_1)} \times (0,\varepsilon) \times U_1 \times \varepsilon^{-1}U_2 \to \mathcal{G}$,

$$\psi_\varepsilon(0,s,y) = (0, (\mathrm{D}_2\psi)(s,y)) \in T_{\phi(s)}\mathcal{G}_{\phi(s)} = A_{\phi(s)}(\mathcal{G})$$

and $\psi_\varepsilon(t,s,y) = (t, \psi(s,ty)) \in (0,1) \times \mathcal{G}$. For ε small enough, the range of ψ_ε will contain v.

The Lie algebroid of $\mathcal{G}_{\mathrm{ad}}$ is the adiabatic Lie algebroid associated to $A(\mathcal{G})$, $A(\mathcal{G}_{\mathrm{ad}}) = A(\mathcal{G})_t$, for all t, such that $\Gamma(A(\mathcal{G})) \cong t\Gamma(A(\mathcal{G} \times [0,\delta)))$.

We expect the above constructions to have applications to semi-classical trace formulæ, see Uribe's overview [47]. A variant of the above example can be used to treat adiabatic limits when the metric is blown up in the base. See [49] for some connections with physics.

More examples are discussed in Section 5.

3. Pseudodifferential operators on groupoids

We proceed now to define the space of pseudodifferential operators acting on sections of vector bundles on a differentiable groupoid. This construction is the same as the one in [32], but slightly more general because we consider also certain non-Hausdorff groupoids. General reference for pseudodifferential operators on smooth manifolds are, for instance, [8] or [45]. We discuss operators on functions, for simplicity, but at the end we briefly indicate the changes necessary to handle operators between sections of smooth vector bundles.

Our construction of pseudodifferential operators on groupoids is obtained considering certain families of pseudodifferential operators on smooth, generally noncompact manifolds. We begin then by recalling a few facts about pseudodifferential operators on smooth manifolds.

Let $W \subset \mathbb{R}^N$ be an open subset. Define the space $\mathcal{S}^m(W \times \mathbb{R}^n)$ of *symbols of order* $m \in \mathbb{R}$ on the bundle $W \times \mathbb{R}^n \to W$, as in [8], to be the set of smooth functions $a : W \times \mathbb{R}^n \to \mathbb{C}$ such that

$$|\partial_y^\alpha \partial_\xi^\beta a(y,\xi)| \leq C_{K,\alpha,\beta}(1 + |\xi|)^{m-|\beta|} \tag{5}$$

for any compact set $K \subset W$ and any multi-indices α and β, and some constant $C_{K,\alpha,\beta} > 0$. A symbol $a \in \mathcal{S}^m(W \times \mathbb{R}^n)$ is called *classical* if it has an asymptotic expansion as an infinite sum of homogeneous symbols $a \sim \sum_{k=0}^\infty a_{m-k}$, a_l homogeneous of degree l for large $\|\xi\|$, i.e. $a_l(y,t\xi) = t^l a_l(y,\xi)$ if $\|\xi\| \geq 1$ and $t \geq 1$. More

precisely, $\sim$ means

$$a - \sum_{k=0}^{M-1} a_{m-k} \in \mathcal{S}^{m-M}(W \times \mathbb{R}^n) \quad \text{for all } M \in \mathbb{N}_0 .$$

The space of classical symbols will be denoted by $\mathcal{S}^m_{\mathrm{cl}}(W \times \mathbb{R}^n)$. Using local trivializations the definition of (classical) symbols immediately extends to arbitrary vector bundles $\pi : E \longrightarrow M$. We shall consider only classical symbols in this paper. For $a \in \mathcal{S}^m(T^*W) = \mathcal{S}^m(W \times \mathbb{R}^n)$ and W an open subset of $\mathbb{R}^n$, we define an operator $a(y, D_y) : \mathcal{C}^\infty_{\mathrm{c}}(W) \to \mathcal{C}^\infty(W)$ by

$$a(y, D_y)u(y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iy\cdot\xi} a(y, \xi)\hat{u}(\xi) d\xi \,, \tag{6}$$

where $\hat{u}$ denotes the Fourier transform of u.

Recall that if M is a smooth manifold, a linear map $T : \mathcal{C}^\infty_{\mathrm{c}}(M) \to \mathcal{C}^\infty(M)$ is called *regularizing* if, and only if, it has a smooth distributional (or Schwartz) kernel. Also, recall that a linear map $P : \mathcal{C}^\infty_{\mathrm{c}}(M) \to \mathcal{C}^\infty(M)$ is called a *(classical) pseudodifferential operator of order* m if, and only if, for all smooth functions ϕ supported in a (not necessarily connected) coordinate chart W, the operator $\phi P \phi$ is of the form $a(y, D_y)$ with a (classical) symbol a of order m. For a classical pseudodifferential operator P as the one considered here, the collection of all classes of a in $\mathcal{S}^m_{\mathrm{cl}}(T^*W)/\mathcal{S}^{m-1}_{\mathrm{cl}}(T^*W)$, for all coordinate neighborhoods W, patches together to define a class $\sigma_m(P) \in \mathcal{S}^m_{\mathrm{cl}}(T^*W)/\mathcal{S}^{m-1}_{\mathrm{cl}}(T^*W)$, which is called *the homogeneous principal symbol* of P; the latter space can, of course, canonically be identified with $S^{[m]}(T^*M)$, the space of all smooth functions $T^*M \setminus \{0\} \longrightarrow \mathbb{C}$ that are positively homogeneous of degree m. We shall sometimes refer to a classical pseudodifferential operator acting on a smooth manifold (without corners) as an *ordinary* classical pseudodifferential operator, in order to distinguish it from a pseudodifferential operator on a groupoid.

We now begin the discussion of pseudodifferential operators on groupoids. A pseudodifferential operator on a differentiable groupoid $\mathcal{G}$ will be a family (P_x), $x \in M$, of classical pseudodifferential operators $P_x : \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x) \to \mathcal{C}^\infty(\mathcal{G}_x)$ with certain additional properties that need to be explained.

If (P_x), $x \in M$, is a family of pseudodifferential operators acting on $\mathcal{G}_x$, we denote by k_x the distribution kernel of P_x We then define the support of the operator P to be

$$\operatorname{supp}(P) = \overline{\bigcup_{x \in M} \operatorname{supp}(k_x)}. \tag{7}$$

The support of P is contained in the closed subset $\{(g, g'), d(g) = d(g')\}$ of the product $\mathcal{G} \times \mathcal{G}$.

To define our class of pseudodifferential operators, we shall need various conditions on the support of our operators. We introduce the following terminology: a family $P = (P_x)$, $x \in M$ is *properly supported* if $p_i^{-1}(K) \cap \operatorname{supp}(P)$ is a compact set for any compact subset $K \subset \mathcal{G}$, where $p_1, p_2 : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ are the two projections. The family $P = (P_x)$ is called *compactly supported* if its support $\operatorname{supp}(P)$ is compact. Finally, P is called *uniformly supported* if its *reduced support* $\operatorname{supp}_\mu(P) := \mu_1(\operatorname{supp}(P))$ is a compact subset of $\mathcal{G}$, where $\mu_1(g', g) := g'g^{-1}$. Clearly, a uniformly supported operator is properly supported, and a compactly supported operator is uniformly supported. If the family $P = (P_x)\,, x \in M$, is

properly supported, then each P_x is properly supported, but the converse is not true.

Recall that the composition of two ordinary pseudodifferential operators is defined if one of them is properly supported. It follows that we can define the composition PQ of two properly supported families of operators $P = (P_x)$ and $Q = (Q_x)$ acting on the fibers of $d : \mathcal{G} \to M$ by pointwise composition $PQ = (P_x Q_x)$, $x \in M$. The resulting family PQ will also be properly supported. If P and Q are uniformly supported, then PQ will also be uniformly supported.

The action of a family $P = (P_x)$ on functions on $\mathcal{G}$ is defined pointwise as follows. For any smooth function $f \in \mathcal{C}^\infty_{\rm c}(\mathcal{G})$ denote by f_x its restriction $f|_{\mathcal{G}_x}$. If each f_x has compact support, and $P = (P_x)\,, x \in M$, is a family of ordinary pseudodifferential operators, then we define Pf by $(Pf)_x = P_x(f_x)$. If P is uniformly supported, then Pf is also compactly supported. However, it is not true that $Pf \in \mathcal{C}^\infty_{\rm c}(\mathcal{G})$ if $f \in \mathcal{C}^\infty_{\rm c}(\mathcal{G})$, in general, so we need some conditions on the family P. We shall hence consider uniformly supported families $P = (P_x)$ because this guarantees that Pf has compact support if f does.

A *fiber preserving diffeomorphism* will be a diffeomorphism $\psi : d(V) \times W \to V$ satisfying $d(\psi(x,w)) = x$, where W is some open subset of an Euclidean space of the appropriate dimension. We now discuss the differentiability condition on a family $P = (P_x)$, a condition which, when satisfied, implies that Pf is smooth for all smooth $f \in \mathcal{C}^\infty_{\rm c}(\mathcal{G})$.

Definition 6. *Let $\mathcal{G}$ be a differentiable groupoid with units M. A family (P_x) of pseudodifferential operators acting on $\mathcal{C}^\infty_c(\mathcal{G}_x)$, $x \in M$, is called* <u>*differentiable*</u> *if, and only if, for any fiber preserving diffeomorphism $\psi : d(V) \times W \to V$ onto an open set $V \subseteq \mathcal{G}$, and for any $\phi \in \mathcal{C}^\infty_c(V)$, we can find $a \in \mathcal{S}^m_{\rm cl}(d(V) \times T^*W)$ such that $\phi P_x \phi$ corresponds to $a(x,y,D_y)$ under the diffeomorphism $\mathcal{G}_x \cap V \simeq W$, for each $x \in d(V)$.*

Thus, we require that the operators P_x be given in local coordinates by symbols a_x that depend smoothly on all variables. Note that nowhere in the above definition it is necessary for $\mathcal{G}$ to be Hausdorff. All we do need is that each of $\mathcal{G}_x$ and $M = \mathcal{G}^{(0)}$ are Hausdorff.

To define pseudodifferential operators on $\mathcal{G}$ we shall consider smooth, uniformly supported families $P = (P_x)$ that satisfy also an invariance condition. To introduce this invariance condition, observe that right translations on $\mathcal{G}$ define linear isomorphisms

$$U_g : \mathcal{C}^\infty(\mathcal{G}_{d(g)}) \to \mathcal{C}^\infty(\mathcal{G}_{r(g)}) : (U_g f)(g') = f(g'g)\,. \tag{8}$$

A family of operators $P = (P_x)$ is then called *invariant* if $P_{r(g)} U_g = U_g P_{d(g)}$, for all $g \in \mathcal{G}$. We are now ready to define pseudodifferential operators on $\mathcal{G}$.

Definition 7. *Let $\mathcal{G}$ be a differentiable groupoid with units M, and let $P = (P_x)$ be a family $P_x : \mathcal{C}^\infty_c(\mathcal{G}_x) \to \mathcal{C}^\infty(\mathcal{G}_x)$ of (order $m \in \mathbb{R}$, ordinary) classical pseudodifferential operators. Then P is an (order m)* <u>*pseudodifferential operator on $\mathcal{G}$*</u> *if, and only if, it is*

(i) uniformly supported,
(ii) differentiable, and
(iii) invariant.

We denote the space of order m pseudodifferential operators on $\mathcal{G}$ by $\Psi^m(\mathcal{G})$.

We also denote $\Psi^{\infty}(\mathcal{G}) := \cup_{m\in\mathbb{R}}\Psi^m(\mathcal{G})$ and $\Psi^{-\infty}(\mathcal{G}) = \cap_{m\in\mathbb{R}}\Psi^m(\mathcal{G})$. Let us now give an alternative description of $\Psi^m(\mathcal{G})$ that highlights again the conormal nature of kernels of pseudodifferential operators. For $P \in \Psi^{\infty}(\mathcal{G})$, we call

$$\kappa_P(g) := k_{d(g)}(g, d(g))\,, g \in \mathcal{G}$$

the *reduced* or *convolution kernel* of P. Due to the right-invariance of P we expect the reduced kernel to carry all information of the family P. For the definition of conormal distributions we refer the reader to [7, 8] in the smooth case.

Proposition 1. *The map $P \longmapsto \kappa_P$ induces an isomorphism*

$$\Psi^m(\mathcal{G}) \xrightarrow{\cong} I^m_c(\mathcal{G}, M; d^*\mathcal{D})$$

where $I^m_c(\mathcal{G}, M; d^*\mathcal{D})$ denotes the space of all compactly supported, $d^*\mathcal{D}$-valued distributions on $\mathcal{G}$ conormal to M. In particular, $P \longmapsto \kappa_P$ identifies $\Psi^{-\infty}(\mathcal{G})$ with the convolution algebra $\mathcal{C}^{\infty}_c(\mathcal{G}, d^*\mathcal{D})$. Moreover, we have $supp(\kappa_P) = supp_\mu(P)$.

Define $\mathcal{C}^{\infty}_{\mathrm{c}}(\mathcal{G})$ as in Section 1, then $\Psi^m(\mathcal{G})(\mathcal{C}^{\infty}_{\mathrm{c}}(\mathcal{G})) \subset \mathcal{C}^{\infty}_{\mathrm{c}}(\mathcal{G})$. We obtain in this way a representation π of $\Psi^{\infty}(\mathcal{G})$ on $\mathcal{C}^{\infty}_{\mathrm{c}}(M)$, uniquely determined by

$$(\pi(P)f) \circ r = P(f \circ r), \quad P = (P_x) \in \Psi^m(\mathcal{G}). \tag{9}$$

We call this representation acting on any space of functions on M on which it makes sense ($\mathcal{C}^{\infty}_{\mathrm{c}}(M)$, $\mathcal{C}^{\infty}(M)$, $L^2(M)$, or Sobolev spaces) the *vector representation* of $\Psi^{\infty}(\mathcal{G})$.

We now discuss the extension of the principal symbol map to $\Psi^m(\mathcal{G})$. Denote by $\pi : A^*(\mathcal{G}) \to M$ the projection. If $P = (P_x) \in \Psi^m(\mathcal{G})$ is an order m pseudodifferential operator on $\mathcal{G}$, then the principal symbol $\sigma_m(P)$ of P will be an order m homogeneous function on $A^*(\mathcal{G}) \smallsetminus 0$ (it is defined only outside the zero section) such that:

$$\sigma_m(P)(\xi) = \sigma_m(P_x)(\xi) \in \mathbb{C} \quad \text{ if } \xi \in A^*_x(\mathcal{G}) = T^*_x\mathcal{G}_x. \tag{10}$$

Denote by $\mathcal{S}^m_c(A^*_x(\mathcal{G})) \subset \mathcal{S}^m_{\mathrm{cl}}(A^*_x(\mathcal{G}))$ the subspace of classical symbol whose support has compact projection onto the space of units M. The above definition determines a linear map

$$\sigma_m : \Psi^m(\mathcal{G}) \to \mathcal{S}^m_c(A^*(\mathcal{G}))/\mathcal{S}^{m-1}_c(A^*(\mathcal{G})) \cong S^{[m]}_c(A^*(\mathcal{G}))\,,$$

where $S^{[m]}_c(A^*(\mathcal{G}))$ denotes the space of all smooth functions $A^*(\mathcal{G}) \setminus \{0\} \longrightarrow \mathbb{C}$ that are positively homogeneous of degree m, and whose support has compact projection onto the space of units M. The map σ_m is said to be the *principal symbol*. A pseudodifferential operator $P \in \Psi^m(\mathcal{G})$ is called *elliptic* provided its principal symbol $\sigma_m(P) \in S^{[m]}_c(A^*(\mathcal{G}))$ does not vanish on $A^*(\mathcal{G}) \setminus \{0\}$; note, with this definition elliptic operators only exist if the space of units is compact.

The following result extends several of the well-known properties of the calculus of ordinary pseudodifferential operators on smooth manifolds to $\Psi^{\infty}(\mathcal{G})$. Denote by $\{\ ,\ \}$ the canonical Poisson bracket on $A^*(\mathcal{G})$.

Theorem 1. *Let $\mathcal{G}$ be a differentiable groupoid. Then $\Psi^{\infty}(\mathcal{G})$ is an algebra with the following properties:*

(i) The principal symbol map

$$\sigma_m : \Psi^m(\mathcal{G}) \to \mathcal{S}^m_c(A^*(\mathcal{G}))/\mathcal{S}^{m-1}_c(A^*(\mathcal{G}))$$

is surjective with kernel $\Psi^{m-1}(\mathcal{G})$.

(ii) If $P \in \Psi^m(\mathcal{G})$ and $Q \in \Psi^{m'}(\mathcal{G})$, then $PQ \in \Psi^{m+m'}(\mathcal{G})$ and satisfies $\sigma_{m+m'}(PQ) = \sigma_m(P)\sigma_{m'}(Q)$. Consequently, $[P, Q] \in \Psi^{m+m'-1}(\mathcal{G})$. Its principal symbol is given by $\sigma_{m+m'-1}([P, Q]) = \frac{1}{i}\{\sigma_m(P), \sigma_{m'}(Q)\}$.

Properly supported invariant differentiable families of pseudodifferential operators also form a filtered algebra, denoted $\Psi^{\infty}_{\mathrm{prop}}(\mathcal{G})$. While it is clear that, in order for our class of pseudodifferential operators to form an algebra, we need some condition on the support of their distributional kernels, exactly what support condition to impose is a matter of choice. We prefer the uniform support condition because it leads to a better control at infinity of the family of operators $P = (P_x)$ and allows us to identify the regularizing ideal (*i.e.,* the ideal of order $-\infty$ operators) with the groupoid convolution algebra of $\mathcal{G}$. The choice of uniform support will also ensure that $\Psi^m(\mathcal{G})$ behaves functorially with respect to open embeddings. The compact support condition enjoys the same properties but is usually too restrictive. The issue of support will be discussed again in examples.

We now discuss the restriction of families in $\Psi^m(\mathcal{G})$ to invariant subsets of M, or, more precisely, the restriction to $\mathcal{G}_Y$, the "reduction" of $\mathcal{G}$ to the invariant subset Y of M. The resulting restriction morphisms $\mathcal{R}_Y : \Psi^\infty(\mathcal{G}) \to \Psi^\infty(\mathcal{G}_Y)$ is the analog in our setting of the indicial morphisms considered in [24].

We continue to denote by $\mathcal{G}$ a differentiable groupoid with units M. Let $Y \subset M$ and let $\mathcal{G}_Y := d^{-1}(Y) \cap r^{-1}(Y)$. Then $\mathcal{G}_Y$ is a groupoid with units Y, called the *reduction of* $\mathcal{G}$ *to* Y. An *invariant* subset $Y \subset M$ is a subset such that $d(g) \in Y$ implies $r(g) \in Y$. For an invariant subset $Y \subset M$, the reduction of $\mathcal{G}$ to Y satisfies

$$\mathcal{G}_Y = d^{-1}(Y) = r^{-1}(Y)$$

and is a differentiable groupoid, if Y is a closed submanifold (with corners) of M.

If $P = (P_x)$, $x \in M$, is a pseudodifferential operator on $\mathcal{G}$, and $Y \subset M$ is an closed, invariant submanifold with corners, we can restrict P to $d^{-1}(Y)$ and obtain

$$\mathcal{R}_Y(P) := (P_x)_{x \in Y},$$

which is a family of operators acting on the fibers of $d : \mathcal{G}_Y = d^{-1}(Y) \to Y$ and satisfying all the conditions necessary to define an element of $\Psi^\infty(\mathcal{G}|_Y)$. This leads to a map

$$\mathcal{R}_Y = \mathcal{R}_{Y,M} : \Psi^\infty(\mathcal{G}) \to \Psi^\infty(\mathcal{G}|_Y), \tag{11}$$

which is easily seen to be an algebra morphism.

Let us indicate now what changes need to be made when we consider operators acting on sections of a vector bundles. Because operators acting between sections of two *different* vector bundles E_1 and E_2 can be recovered from operators acting on $E = E_1 \oplus E_2$, we may assume that $E_1 = E_2 = E$ as vector bundles on $M = \mathcal{G}^{(0)}$.

Denote by $r^*(E)$ the pull-back of E to $\mathcal{G} = \mathcal{G}^{(1)}$. Then the isomorphisms of Equation (8) will have to be replaced by

$$U_g : \mathcal{C}^\infty(\mathcal{G}_{d(g)}, r^*(E)) \to \mathcal{C}^\infty(\mathcal{G}_{r(g)}, r^*(E)) : (U_g f)(g') = f(g'g) \in (r^*E)_{g'},$$

which makes sense because of $(r^*E)_{g'} = (r^*E)_{g'g} = E_{r(g')}$. Then, to define $\Psi^m(\mathcal{G}; E)$ we consider families $P = (P_x)$ of order m pseudodifferential operators P_x, $x \in M$, acting on the spaces $\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}_x, r^*(E))$. We require these families to be uniformly supported, differentiable, and invariant, as in the case $E = \mathbb{C}$.

The principal symbol $\sigma_m(P)$ of a classical pseudodifferential operator P belongs then to $\mathcal{S}_c^m(T^*W;\mathrm{End}(E))/\mathcal{S}_c^{m-1}(T^*W;\mathrm{End}(E))$. Finally, the restriction (or indicial) morphism is a morphism.

$$\mathcal{R}_Y : \Psi^\infty(\mathcal{G};E) \to \Psi^\infty(\mathcal{G}|_Y;E|_Y).$$

All the other changes needed to treat the case of non-trivial vector bundles E are similar.

There is one particular case of a bundle E that deserves special attention. Let $E := \mathcal{D}^{1/2}$ be the square root of the density bundle $\mathcal{D} = |\Lambda^n A(\mathcal{G})|$, as before. If $P \in \Psi^m(\mathcal{G};\mathcal{D}^{1/2})$ consists of the family $(P_x, x \in M)$, then each P_x acts on $V_x = C_c^\infty(\mathcal{G}_x; r^*(\mathcal{D}^{1/2}))$. Since $r^*(\mathcal{D}^{1/2}) = \Omega^{1/2}_{\mathcal{G}_x}$ is the bundle of half densities on $\mathcal{G}_x$, we can define a natural hermitian inner product $(\ ,\ )$ on V_x. Let $P = (P_x) \in \Psi^m(\mathcal{G};\mathcal{D}^{1/2})$, and denote by P_x^* the formal adjoint of P_x, that is, the unique pseudodifferential operator on V_x such that $(P_x f, g) = (f, P_x^* g)$, for all $f, g \in V_x$. It is not hard to see that $(P_x^*) \in \Psi^m(\mathcal{G};\mathcal{D}^{1/2})$. Moreover, $\sigma_m(P^*) = \overline{\sigma_m(P)}$.

If E is the complexification of a real bundle E_0: $E \simeq E_0 \otimes \mathbb{C}$, then the complex conjugation operator $J \in \mathrm{End}_{\mathbb{R}}(E)$ defines a real structure on $\Psi^*(\mathcal{G};E)$, that is, a conjugate linear involution on $\Psi^*(\mathcal{G};E)$. In this case, $\Psi^*(\mathcal{G};E)$ is the complexification of the set of fixed points of this involution.

4. Bounded representations

For a smooth, compact manifold M (without corners), the algebra $\Psi^0(M)$ of order zero pseudodifferential operators on M acts by bounded operators on $L^2(M, d\mu)$, the Hilbert space $L^2(M, d\mu)$ being defined with respect to the (essentially unique) measure μ corresponding to a nowhere vanishing density on M. Moreover, this is basically the only interesting $*$-representation of $\Psi^0(M)$ by bounded operators on an infinite-dimensional Hilbert space of functions.

For a differentiable groupoid $\mathcal{G}$ with units M, a manifold with corners, it is still true that we can find a measure μ such that $\Psi^0(\mathcal{G})$ acts by bounded operators on $L^2(M, d\mu)$. However, in this case there may exist natural measures $d\mu$ that are singular with respect to the measure defined by a nowhere vanishing density. Moreover, there may exist several non-equivalent such measures, and these representations may not exhaust all equivalence classes of non-trivial, irreducible, bounded representations of $\Psi^0(\mathcal{G})$.

The purpose of this section is to introduce the class of representations we are interested in, and to study some of their properties. A consequence of our results is that in order to construct and classify bounded representations of $\Psi^0(\mathcal{G})$, it is essentially enough to do this for $\Psi^{-\infty}(\mathcal{G})$.

We are interested in representations of $\Psi^m(\mathcal{G})$, $m \in \{0, \pm\infty\}$. We fix a trivialization of $\mathcal{D}$, so that we get an isomorphism $\Psi^m(\mathcal{G}) \cong \Psi^m(\mathcal{G};\mathcal{D}^{1/2})$ and hence we have an involution $*$ on $\Psi^m(\mathcal{G})$. Let $\mathcal{H}_0$ be a dense subspace of a Hilbert space $\mathcal{H}$ with inner product $(\ ,\ )$. Recall that a $*$-morphism $\alpha : A \to \mathrm{End}(\mathcal{H}_0)$ from a $*$-algebra A is a morphism such that $(\alpha(P^*)\xi, \eta) = (\xi, \alpha(P)\eta))$, for all $P \in A$ and all $\xi, \eta \in \mathcal{H}_0$.

Definition 8. *Let $\mathcal{H}_0$ be a dense subspace of a Hilbert space $\mathcal{H}$, and $m = 0$ or $m = \pm\infty$ be fixed. A bounded representation of $\Psi^m(\mathcal{G})$ on $\mathcal{H}_0$ is a $*$-morphism $\varrho : \Psi^m(\mathcal{G}) \to \mathrm{End}(\mathcal{H}_0)$ such that $\varrho(P)$ extends to a bounded operator on $\mathcal{H}$ for all $P \in \Psi^0(\mathcal{G})$ (for all $P \in \Psi^{-\infty}(\mathcal{G})$, if $m = -\infty$).*

Note that if ϱ is as above and P is an operator of positive order, then $\varrho(P)$ does not have to be bounded. Since $\mathcal{H}_0$ is dense in $\mathcal{H}$, each operator $\varrho(P)$ can be regarded as a densely defined operator. The definition implies that $\varrho(P^*) \subset \varrho(P)^*$, so the adjoint of $\varrho(P)$ is densely defined, and hence $\varrho(P)$ is a closable operator. We shall usually make no distinction between $\varrho(P)$ and its closure.

We call a bounded representation $\varrho : \Psi^{-\infty}(\mathcal{G}) \to \mathrm{End}(\mathcal{H})$ *non-degenerate* provided $\varrho(\Psi^{-\infty}(\mathcal{G}))\mathcal{H}$ is dense in $\mathcal{H}$.

The following theorem establishes, among other things, a bijective correspondence between non-degenerate bounded representations of $\Psi^{-\infty}(\mathcal{G})$ on a Hilbert space $\mathcal{H}$, and bounded representations $\varrho : \Psi^{\infty}(\mathcal{G}) \to \mathrm{End}(\mathcal{H}_0)$ such that the space $\varrho(\Psi^{-\infty}(\mathcal{G}))\mathcal{H}_0$ is dense in $\mathcal{H}$. We shall need the following slight extension of a result in [13].

Theorem 2. *Let $\mathcal{H}$ be a Hilbert space and let $\varrho : \Psi^{-\infty}(\mathcal{G}) \to \mathrm{End}(\mathcal{H})$ be a bounded representation. Then, to each $P \in \Psi^s(\mathcal{G})$, $s \in \mathbb{R}$, we can associate an unbounded operator $\varrho(P)$, with domain $\mathcal{H}_0 := \varrho(\Psi^{-\infty}(\mathcal{G}))\mathcal{H}$, such that $\varrho(P)\varrho(R) = \varrho(PR)$ and $\varrho(R)\varrho(P) = \varrho(RP)$, for any $R \in \Psi^{-\infty}(\mathcal{G})$.*

We obtain in this way an extension of ϱ to a bounded representation of $\Psi^0(\mathcal{G})$ on $\mathcal{H}$ and to a bounded representation of $\Psi^{\infty}(\mathcal{G})$ on $\mathcal{H}_0 := \varrho(\Psi^{-\infty}(\mathcal{G}))\mathcal{H}$.

Proof. We may assume that $\mathcal{H}_0$ is dense in $\mathcal{H}$. Fix $P \in \Psi^0(\mathcal{G})$ and we let

$$\varrho(P)\xi = \varrho(PQ)\eta,$$

if $\xi = \varrho(Q)\eta$, for some $Q \in \Psi^{-\infty}(\mathcal{G})$ and $\eta \in \mathcal{H}$. We need to show that $\varrho(P)$ is well-defined and bounded. Thus, we need to prove that $\sum_{k=1}^{N} \varrho(PQ_k)\xi_k = 0$, if $P \in \Psi^0(\mathcal{G})$ and $\sum_{k=1}^{N} \varrho(Q_k)\xi_k = 0$, for some $Q_k \in \Psi^{-\infty}(\mathcal{G})$ and $\xi_k \in \mathcal{H}$.

We will show that, for each $P \in \Psi^0(\mathcal{G})$, there exists a constant $k_P > 0$ such that

$$\|\sum_{k=1}^{N} \varrho(PQ_k)\xi_k\| \leq k_P \|\sum_{k=1}^{N} \varrho(Q_k)\xi_k\|. \tag{12}$$

Let $C \geq |\sigma_0(P)| + 1$, and let

$$b = (C^2 - |\sigma_0(P)|^2)^{1/2}. \tag{13}$$

Then $b - C$ is in $\mathcal{C}_{\mathrm{c}}^{\infty}(S^*(\mathcal{G}))$, and it follows from Theorem 1 that we can find $Q_0 \in \Psi^0(\mathcal{G})$ such that $\sigma_0(Q_0) = b - C$. Let $Q = Q_0 + C$. Using again Theorem 1, we obtain, for

$$R = C^2 - P^*P - Q^*Q \in \Psi^0(\mathcal{G}),$$

that

$$\sigma_0(R) = \sigma_0(C^2 - P^*P - Q^*Q) = 0,$$

and hence $R \in \Psi^{-1}(\mathcal{G})$. A standard argument using the asymptotic completeness of the algebra of pseudodifferential operators shows that we can assume that Q has order $-\infty$. Let then

$$\xi = \sum_{k=1}^{N} \varrho(Q_k)\xi_k, \quad \eta = \sum_{k=1}^{N} \varrho(PQ_k)\xi_k, \quad \text{and} \quad \zeta = \sum_{k=1}^{N} \varrho(QQ_k)\xi_k, \tag{14}$$

which gives,

$$\|\eta\|^2 = (\eta,\eta) = \sum_{j,k=1}^{N} (\varrho(Q_k^* P^* P Q_j)\xi_j, \xi_k) \tag{15}$$

$$= \sum_{j,k=1}^{N} \big(C^2(\varrho(Q_k Q_j)\xi_j,\xi_k) - (\varrho(Q_k^* Q^* Q Q_j)\xi_j,\xi_k) - (\varrho(Q_k^* R Q_j)\xi_j,\xi_k)\big)$$

$$= C^2\|\xi\|^2 - \|\zeta\|^2 - (\varrho(R)\xi,\xi) \le (C^2 + \|\varrho(R)\|)\|\xi\|^2.$$

The desired representation of $\Psi^0(\mathcal{G})$ on $\mathcal{H}$ is obtained by extending $\varrho(P)$ by continuity to $\mathcal{H}$.

To extend ϱ further to $\Psi^s(\mathcal{G})$, we proceed similarly: we want

$$\varrho(P)\varrho(Q)\xi = \varrho(PQ)\xi,$$

for $P \in \Psi^\infty(\mathcal{G})$ and $Q \in \Psi^{-\infty}(\mathcal{G})$. Let ξ and η be as in Equation (14). We need to prove that $\eta = 0$ if $\xi = 0$. Now, because $\mathcal{H}_0$ is dense in $\mathcal{H}$, we can find T_j in A_ϱ the norm closure of $\varrho(\Psi^{-\infty}(\mathcal{G}))$ and $\eta_j \in \mathcal{H}$ such that $\eta = \sum_{j=1}^N T_j\eta_j$. Choose an approximate unit u_α of the C^*-algebra A_ϱ, then $u_\alpha T_j \to T_j$ (in the sense of generalized sequences). We can replace then the generalized sequence (net) u_α by a subsequence, call it u_m such that $u_m T_j \to T_j$, as $m \to \infty$. By density, we may assume $u_m = \varrho(R_m)$, for some $R_m \in \Psi^{-\infty}(\mathcal{G})$. Consequently, $\varrho(R_m)\eta \to \eta$, as $m \to \infty$. Then

$$\eta = \lim \sum_{k=1}^{N} \varrho(R_m)\varrho(PQ_k)\xi_k = \lim \sum_{k=1}^{N} \varrho(R_m P)\varrho(Q_k)\xi_k = 0,$$

because $R_m P \in \Psi^{-\infty}(\mathcal{G})$. □

Remark. We also obtain, using the above notation, that any extension of ϱ to a representation of $\Psi^0(\mathcal{G})$ is bounded. This extension is uniquely determined if $\mathcal{H}_0$ is dense in $\mathcal{H}$.

Assume that M is connected, so that the manifolds $\mathcal{G}_x$ have the same dimension n. We now proceed to define a Banach norm on $\Psi^{-n-1}(\mathcal{G})$. This norm depends on the choice of a trivialization of the bundle of densities $\mathcal{D}$, which then gives rise to a right invariant system of measures μ_x. Then, if $P \in \Psi^{-n-1}(\mathcal{G})$, we use the chosen trivialization of $\mathcal{D}$ to identify the reduced kernel κ_P, which is a priori a distribution, with a compactly supported, *continuous function* on $\mathcal{G}$, still denoted by κ_P. We then define

$$\|P\|_1 = \sup_{x\in M} \left\{ \int_{\mathcal{G}_x} |\kappa_P(g^{-1})| d\mu_x(g), \int_{\mathcal{G}_x} |\kappa_P(g)| d\mu_x(g) \right\}. \tag{16}$$

Some of the most interesting representations of $\Psi^\infty(\mathcal{G})$ are the regular representations π_x, $x \in M$. These are bounded representations defined as follows; let $x \in M$, then the *regular representation* π_x associated to x is the natural representation of $\Psi^\infty(\mathcal{G})$ on $C_c^\infty(\mathcal{G}_x; r^*(\mathcal{D}^{1/2}))$, that is $\pi_x(P) = P_x$. Moreover, $\|\pi_x(P)\| \le \|P\|_1$, if $P \in \Psi^{-n-1}(\mathcal{G})$.

Define now the *reduced norm* of P by

$$\|P\|_r = \sup_{x\in M} \|\pi_x(P)\| = \sup_{x\in M} \|P_x\|,$$

and the *full norm* of P by

$$\|P\| = \sup_{\varrho} \|\varrho(P)\|,$$

where ϱ ranges through all bounded representations ϱ of $\Psi^0(\mathcal{G})$ satisfying

$$\|\varrho(P)\| \leq \|P\|_1 \quad \text{for all } P \in \Psi^{-\infty}(\mathcal{G})\,.$$

The above comments imply, in particular, that $\|P\|_r \leq \|P\|$. If we have equality, we shall call $\mathcal{G}$ *amenable*, following the standard usage.

Denote by $\mathfrak{A}(\mathcal{G})$ [respectively, by $\mathfrak{A}_r(\mathcal{G})$] the closure of $\Psi^0(\mathcal{G})$ in the norm $\| \ \|$ [respectively, in the norm $\| \ \|_r$]. Also, denote by $C^*(\mathcal{G})$ [respectively, by $C^*_r(\mathcal{G})$] the closure of $\Psi^{-\infty}(\mathcal{G})$ in the norm $\| \ \|$ [respectively, in the norm $\| \ \|_r$]. The principal symbol σ_0 extends by continuity to $\mathfrak{A}(\mathcal{G})$ and $\mathfrak{A}_r(\mathcal{G})$.

Let $S^*(\mathcal{G}) := (A^*(\mathcal{G}) \setminus \{0\})/\mathbb{R}_+$ be the space of rays in $A^*(\mathcal{G})$. (By choosing a metric on $A(\mathcal{G})$, we may identify $S^*(\mathcal{G})$ with the subset of vectors of length one in $A^*(\mathcal{G})$.) Then we have the following two exact sequences of C^*-algebras:

$$\begin{gathered} 0 \to C^*(\mathcal{G}) \to \mathfrak{A}(\mathcal{G}) \to \mathcal{C}_0(S^*(\mathcal{G})) \to 0 \quad \text{and} \\ 0 \to C^*_r(\mathcal{G}) \to \mathfrak{A}_r(\mathcal{G}) \to \mathcal{C}_0(S^*(\mathcal{G})) \to 0\,. \end{gathered}$$

In particular, $\Psi^{-\infty}(\mathcal{G})$ is dense in $\Psi^{-1}(\mathcal{G})$.

Let $Y \subset M$ be a closed, invariant submanifold with corners. Then we also have exact sequences

$$0 \to C^*(\mathcal{G}_{M\setminus Y}) \to C^*(\mathcal{G}) \to C^*(\mathcal{G}_Y) \to 0 \quad \text{and} \tag{17}$$

$$0 \to \mathfrak{A}(\mathcal{G}_{M\setminus Y}) \to \mathfrak{A}(\mathcal{G}) \to \mathfrak{A}(\mathcal{G}_Y) \to 0. \tag{18}$$

(We will not use that, but it is interesting to mentioned that it is known that there are no such exact sequence for reduced C^*-algebras, in general.)

All the morphisms of the above four exact sequences are compatible with the complex conjugation on these algebras.

Definition 9. *An invariant filtration $Y_0 \subset Y_1 \subset \cdots \subset Y_n = M$ is an increasing sequence of closed, invariant subsets of M with the property that the closure $\overline{S}$ of each connected component S of $Y_k \smallsetminus Y_{k-1}$ is a submanifold with corners of M and that $\overline{S} \smallsetminus S$ is the union of the hyperfaces of $\overline{S}$ (that is, $S = \overline{S} \smallsetminus \partial S$).*

The exact sequences defined before then give the following result:

Theorem 3. *Let $\mathcal{G}$ be a differentiable groupoid with space of units M, and let $\emptyset =: Y_{-1} \subset Y_0 \subset Y_1 \subset \cdots \subset Y_n = M$ be an invariant filtration of M. Define $\mathfrak{I}_k := C^*(\mathcal{G}_{M \smallsetminus Y_{k-1}})$. Then we get a composition series of closed ideals*

$$(0) \subset \mathfrak{I}_n \subset \mathfrak{I}_{n-1} \subset \ldots \subset \mathfrak{I}_0 = C^*(\mathcal{G}) \subset \mathfrak{A}(\mathcal{G})\,,$$

whose subquotients are determined by

$$\begin{aligned} \sigma_0 : \mathfrak{A}(\mathcal{G})/\mathfrak{I}_0 &\xrightarrow{\cong} \mathcal{C}_0(S^*\mathcal{G})\,, \text{ and} \\ \mathfrak{I}_k/\mathfrak{I}_{k+1} &\cong C^*(\mathcal{G}_{Y_k \smallsetminus Y_{k-1}}) \quad \text{for } 0 \leq k \leq n\,. \end{aligned}$$

A completely analogous result holds for the norm closure of the algebra $\Psi^0(\mathcal{G}; E)$, for any Hermitian vector bundle E. In fact, we can find an orthogonal projection $p_E \in M_N(\mathcal{C}^\infty(M))$, for some large N, such that $E \cong p_E(M \times \mathbb{C}^N)$, and hence $\Psi^0(\mathcal{G}; E) \cong p_E M_N(\Psi^0(\mathcal{G})) p_E$.

The definition of an invariant filtration given in this paper is slightly more general than the one in [13], however, these definitions are equivalent if each $\mathcal{G}_x$ is connected.

Thus, in order to avoid some unnecessary technical complications, *we shall assume from now on that all the fibers* $\mathcal{G}_x := d^{-1}(x)$ *of* d *are connected.* (Recall that a groupoid with this property is called *d-connected.*)

We observe then, that if $(Y_k)_k$ is an invariant filtration, then each connected component of $Y_k \setminus Y_{k-1}$ is an invariant subset of M and

$$C^*(\mathcal{G}_{Y_k \setminus Y_{k-1}}) \cong \bigoplus_S C^*(\mathcal{G}_S)$$

where S ranges through the set of open components of $Y_k \setminus Y_{k-1}$. Moreover, a completely similar direct sum decomposition exists for $\mathfrak{A}(\mathcal{G}_{Y_k \setminus Y_{k-1}})$.

A first consequence of the above theorem is that if each $\mathcal{G}_S$ is an amenable groupoid (that is, $C^*(\mathcal{G}_S) \cong C^*_r(\mathcal{G}_S)$), then $\mathcal{G}$ is also amenable. This can be seen as follows. Using an argument based on induction, it is enough to prove that if a groupoid $\mathcal{G}$ has an open invariant subset $\mathcal{O}$ such that both $\mathcal{G}_{\mathcal{O}}$ and $\mathcal{G}_{M \setminus \mathcal{O}}$ are amenable, then $\mathcal{G}$ is amenable. To prove this, let I be the kernel of the natural map $C^*(\mathcal{G}) \to C^*_r(\mathcal{G})$ which is onto because its range is closed and contains the dense subspace $\Psi^{-\infty}(\mathcal{G})$. Since $\mathcal{G}_{M \setminus \mathcal{O}}$ is amenable, I is in the kernel of the restriction homomorphism $C^*(\mathcal{G}) \to C^*(\mathcal{G}_{M\setminus\mathcal{O}})$, *i.e.,* I is a subset of $C^*(\mathcal{G}_{\mathcal{O}})$. But the maps $C^*(\mathcal{G}_{\mathcal{O}}) \to C^*_r(\mathcal{G}_{\mathcal{O}})$ and $C^*_r(\mathcal{G}_{\mathcal{O}}) \to C^*_r(\mathcal{G})$ are both injective. Hence $I = 0$.

The above theorem leads to a characterization of compactness and Fredholmness for operators in $\Psi^0(\mathcal{G})$. This characterization is similar, and it actually contains as a particular case, the characterization of Fredholm operators in the "b-calculus" or one of its variants on manifolds with corners, see [24]. Characterizations of compact and Fredholm operators on manifolds with singularities were, for instance, also obtained in [14, 18, 20, 22, 33, 34, 43, 44].

The significance of Theorem 3 is that often in practice we can find nice invariant stratifications $M = \bigcup S$ for which the subquotients $C^*(\mathcal{G}_S)$ have a relatively simpler structure than that of $C^*(\mathcal{G})$ itself. An example is the b-calculus and its generalizations, the c_n-calculi, which are discussed in Section 10.

In the following, we shall denote by $\otimes_{min}$ the *minimal* tensor product of C^*–algebras, defined using the tensor product of Hilbert spaces, see [41]. More precisely, assume that A_i, $i = 1, 2$, are C^*-algebras, which we may assume to be closed subalgebras of the algebras of bounded operators on some Hilbert spaces $\mathcal{H}_i$. Then the algebraic tensor product $A_1 \otimes A_2$ acts on (the Hilbert space completion of) $\mathcal{H}_1 \otimes \mathcal{H}_2$, and $A_1 \otimes_{min} A_2$ is defined to be the completion of $A_1 \otimes A_2$ with respect to the induced norm. The following result is sometimes useful.

Proposition 2. *If* $\mathcal{G}_i$, $i = 0, 1$, *are two differential groupoids, then*

$$C^*_r(\mathcal{G}_0 \times \mathcal{G}_1) \simeq C^*_r(\mathcal{G}_0) \otimes_{min} C^*_r(\mathcal{G}_1).$$

5. Examples II: Pseudodifferential Operators

The examples of differentiable groupoids of Section 2 also lead to interesting algebras of pseudodifferential operators. Many well-known algebras of pseudodifferential operators are in fact (isomorphic to) algebras of the form $\Psi^\infty(\mathcal{G})$. This leads to new insight into the structure of these algebras. In additions to these well-known algebras, we also obtain algebras that are difficult to describe directly, without using groupoids. Moreover, some of these algebras were not considered before the groupoids were introduced into the picture, nevertheless, these algebras

are expected to play an important role in the analysis on certain non-compact manifolds.

Our examples will follow in the beginning the same order as the examples considered in Section 2.

Example 11. If $\mathcal{G} = M$ is a manifold (possibly with corners), then we have $\Psi^\infty(\mathcal{G}) \simeq \mathcal{C}_c^\infty(M)$ and $\Psi^\infty_{prop}(\mathcal{G}) = \mathcal{C}^\infty(M)$.

Denote by $\Psi^m_{\mathrm{prop}}(M)$ the space of *properly supported* pseudodifferential operators on a smooth manifold M.

Example 12. If $\mathcal{G} = G$ is a Lie group, then $\Psi^m(\mathcal{G}) \simeq \Psi^m_{\mathrm{prop}}(G)^G$, the algebra of properly supported pseudodifferential operators on G, invariant with respect to right translations. In this example, every invariant properly supported operator is also uniformly supported.

The following example shows that the algebras of pseudodifferential operators (with appropriate support conditions for the Schwartz kernels) on a smooth manifolds without corners can be recovered as algebras of pseudodifferential operators on the pair groupoid. Let $\Psi^m_{\mathrm{comp}}(M)$ be the space of pseudodifferential operators on M with compactly supported Schwartz kernels.

Example 13. Suppose now that $\mathcal{G} = M \times M$, with M a smooth manifold without corners, is the pair groupoid. Then $\Psi^m(\mathcal{G}) \cong \Psi^m_{comp}(M)$ and $\Psi^m_{prop}(\mathcal{G}) \cong \Psi^m_{prop}(M)$.

Moreover, the vector representation of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}_c^\infty(M)$ recovers the usual action of pseudodifferential operators on functions on M. (Recall from (9), that the vector representation π of $\Psi^\infty(\mathcal{G})$ is given by $(\pi(P)f) \circ r = P(f \circ r)$.)

The fibered pair recovers families of operators.

Example 14. If $\mathcal{G} = M \times_B M$ is the fibered pair groupoid, for some submersion $M \to B$, then $\Psi^m(\mathcal{G})$ consists of families of pseudodifferential operators along the fibers of $M \to B$ such that their reduced kernels are compactly supported (as distributions on $\mathcal{G}$).

The vector representation π of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}_c^\infty(M)$ is just the usual action of families of pseudodifferential operators on functions, the action being defined fiberwise.

The following three examples of algebras were probably considered only in the framework of groupoid algebras, although particular cases have been investigated before.

Example 15. For a product groupoid $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ there is no obvious description of $\Psi^\infty(\mathcal{G}_1 \times \mathcal{G}_2)$ in terms of $\Psi^\infty(\mathcal{G}_1)$ and $\Psi^\infty(\mathcal{G}_2)$, in general. However, when $\mathcal{G}_1 = M_1$ is a manifold with corners (so $\mathcal{G}_1$ has no non-trivial arrows), then $\Psi^m(\mathcal{G}_1 \times \mathcal{G}_2)$ consists of families of operators in $\Psi^m(\mathcal{G}_2)$ parameterized by M_1. For smoothing operators the situation is simpler: $\Psi^{-\infty}(\mathcal{G}_1 \times \mathcal{G}_2)$ contains naturally the tensor product $\Psi^{-\infty}(\mathcal{G}_1) \otimes \Psi^{-\infty}(\mathcal{G}_2)$ as a dense subset.

An interesting particular case is when $\mathcal{G}_1 = M \times M$, the pair groupoid, and $\mathcal{G}_2 = \mathbb{R}^q$ (that is, the groupoid associated to the Lie group $\mathbb{R}^q$), then $\Psi^m(\mathcal{G}_1 \times \mathcal{G}_2)$ can be identified with a natural, dense subalgebra of the algebra of q-suspended pseudodifferential operators "on" M, introduced by Melrose.

Example 16. If $\mathcal{G} \to B$ is a bundle of Lie groups, then $\Psi^m(\mathcal{G})$ consists of smooth families of invariant, properly supported, pseudodifferential operators on the fibers

of $\mathcal{G} \to B$. For $\mathcal{G} = B \times G$, a trivial bundle of Lie groups, $\mathcal{G}$ is the product (as groupoids) of a smooth manifold B, as in Example 1, and a Lie group G, as in Example 2. A very important particular case of this construction is when $\mathcal{G} \to B$ is a vector bundle, with the induced fiberwise operations. We shall use this example below several times.

Example 17. Again, the only general thing that can be said about fibered products is that $\Psi^{-\infty}(\mathcal{G}_1) \otimes_{\mathcal{C}^\infty(B)} \Psi^{-\infty}(\mathcal{G}_2)$ identifies with a dense subset of $\Psi^{-\infty}(\mathcal{G}_1 \times_B \mathcal{G}_2)$. When $\mathcal{G}_1 = M \times_B M$ is a fibered pair groupoid and $\mathcal{G}_2$ is a bundle of Lie groups on B, then $\Psi^m(\mathcal{G}_1 \times \mathcal{G}_2)$ is an algebra considered in [30], and consists of smooth families of pseudodifferential operators on $M \times_B \mathcal{G}_2$ invariant with respect to the bundle of Lie groups $\mathcal{G}_2$.

Example 18 (Connes). If $\mathcal{G}$ is the holonomy groupoid associated to the foliated manifold (M, F), then $\Psi^*(\mathcal{G})$ is the algebra of pseudodifferential operators along the leaves of (M, F), considered first by Connes [2]. In fact, our algebra is a little smaller than Connes' who considered families that are only continuous in the transverse direction. These algebras, however, have the same formal properties as our families.

Example 19. Let $\mathcal{G}$ be the fundamental groupoid of a compact smooth manifold M with fundamental group $\pi_1(M) = \Gamma$. If $P = (P_x)_{x \in M} \in \Psi^m(\mathcal{G})$, then each P_x, $x \in M$, is a pseudodifferential operator on $\widetilde{M}$. The invariance condition applied to the elements g such that $x = d(g) = r(g)$ implies that each operator P_x is invariant with respect to the action of Γ. This means that we can identify P_x with an operator on $\widetilde{M}$ and that the resulting operator does not depend on the identification of $\mathcal{G}_x$ with $\widetilde{M}$. Then the invariance condition applied to an arbitrary arrow $g \in \mathcal{G}$ gives that all operators P_x acting on $\widetilde{M}$ coincide. We obtain $\Psi^m(\mathcal{G}) \simeq \Psi^m_{\text{prop}}(\widetilde{M})^\Gamma$, the algebra of properly supported Γ-invariant pseudodifferential operators on the universal covering $\widetilde{M}$ of M. An alternative definition of this algebra using crossed products is given in [28].

Example 20. If $\mathcal{G}_{\text{ad}}$ is the adiabatic groupoid associated to a groupoid $\mathcal{G}$, then an operator $P \in \Psi^m(\mathcal{G}_{\text{ad}})$ consists of a family $P = (P_{t,x})$, $t \geq 0$, $x \in M$, (M is the space of units of $\mathcal{G}$), such that if we denote by P_t the family $(P_{t,x})$, for a fixed t, then $P_t \in \Psi^m(\mathcal{G})$ for $t > 0$ and P_t depends smoothly on t in this range. For $t = 0$, $P_0 \in \Psi^m(A(\mathcal{G}))$, is a family of operators on the fibers of $A(\mathcal{G}) \to M$, translation invariant with respect to the variable in each fiber. Thus, $\Psi^m(A(\mathcal{G}))$ is one of the algebras appearing in Example 16.

In a certain sense $P_t \to P_0$, as $t \to 0$, but this is difficult to make precise without considering the adiabatic groupoid. (Actually, making precise the fact that the family P_t is smooth at 0 also is precisely the *raison d'être* for the algebra of pseudodifferential operators on the adiabatic groupoid.)

The best way to formalize this continuity property is the following. Consider the evaluation morphisms $e_t : \Psi^m(\mathcal{G}_{\text{ad}}) \to \Psi^m(\mathcal{G})$, if $t > 0$, and $e_0 : \Psi^m(\mathcal{G}_{\text{ad}}) \to \Psi^m(A(\mathcal{G}))$. (These morphisms are particular instances of the restriction morphisms defined in Equation (11)). If $P \in \Psi^0(\mathcal{G}_{\text{ad}})$, then $\|e_t(P)\|$ and $\|e_t(P)\|_r$ are continuous in t. This was proved by Landsman and Ramazan, see [10, 11, 36].

Some typical operators in $\Psi^m(\mathcal{G}_{\mathrm{ad}})$ are obtained by rescaling the symbol of a differential operator D on $\mathcal{G}$. To see how this works, note first that there exists a polynomial symbol a on $A^*(\mathcal{G})$ such that $q(a) = D$, where $q : \mathcal{S}^m(A^*(\mathcal{G})) \to \Psi^m(\mathcal{G})$ is the quantization map considered in [32]. Let a_t be the symbol $a_t(\xi) = a(t\xi)$, for $t > 0$, and also let $q_{\mathrm{ad}} : \mathcal{S}^m(A^*(\mathcal{G}_{adb})) \to \Psi^m(\mathcal{G}_{\mathrm{ad}})$ be the quantization map for the adiabatic groupoid. We can extend a to a symbol on $A^*(\mathcal{G}_{\mathrm{ad}})$ constant in t, then $e_t(q_{\mathrm{ad}}(a)) = q(a_t)$, if $t > 0$. For $t = 0$, we obtain that $e_0(q_{\mathrm{ad}}(a))$ is isomorphic to the operator of multiplication by a, after taking the Fourier transform along the fibers of $A^*(\mathcal{G})$.

An important class of examples is obtained by integrating suitable Lie algebras of vector fields on a manifold M with corners. This is related to Melrose's approach to a pseudodifferential analysis on manifolds with corners [21], though our techniques are different in the end. We thus start with a Lie subalgebra $\mathcal{V}$ of the Lie algebra of all vector fields that are tangent to each boundary hyperface of a given manifold M with corners. The Lie algebra $\mathcal{V}$ can be thought of as determining the degeneracies of our operators near the boundary. If $\mathcal{V}$ is in addition a projective $\mathcal{C}^\infty(M)$-module, then, by the Serre-Swan theorem, there is a smooth vector bundle $A = {}^{\mathcal{V}}TM \to M$ together with a smooth map of vector bundles $q : A \longrightarrow TM$ such that $\mathcal{V} = q(\Gamma(A))$. (This will be discussed in more detail in a forthcoming book of Melrose on manifolds with corners.)

The next step is to integrate this Lie algebroid A, that is, to find a Lie groupoid $\mathcal{G}$ with Lie algebroid A. Here, we can follow the general method used in [29]. The integration procedure consists in fact of two steps. Let us denote by A_S the restriction of A to each open boundary face S of M of positive codimension, suppose that we can find differentiable groupoids $\mathcal{G}_S$ integrating A_S, and let $\mathcal{G} = \bigcup \mathcal{G}_S$. By [29], there exists at most one smooth structure on $\mathcal{G}$ compatible with the groupoid operations. Whenever such a smooth structure exists, the resulting groupoid satisfies $A(\mathcal{G}) = A$. Moreover, if the $\mathcal{G}_S$ are maximal among all d-connected groupoids integrating A_S, then there is a natural differentiable structure on $\mathcal{G}$ making it into a differentiable groupoid with Lie algebroid A. Note that this choice for $\mathcal{G}$ will almost always lead us to non-Hausdorff groupoids and to problems related to the analysis on these spaces. Moreover, the vector representation will not be injective, in general. The reason is that the maximal d-connected groupoid integrating a given Lie algebroid is much to big. For instance, for the Lie algebroid $TM \to M$, the maximal d-connected groupoid integrating it is the path groupoid [32], not the pair groupoid as expected and usually desired [17]. In particular cases, however, the given Lie algebroid A can be integrated directly to a Hausdorff differentiable groupoid. These remarks apply to the following two examples. These two examples are essentially due to Melrose [21] and, respectively, to Mazzeo [19]. A groupoid for a special case of Example 21 (b-calculus on manifolds with corners) was constructed in [26], see also [32].

Example 21. *The "very small" c_n-calculus.* Let M be a compact manifold with corners, and associate to each hypersurface $H \subset M$ an integer $c_H \geq 1$. We also fix a defining function for each hypersurface. Choose also on M a metric h such that each point $p \in F$, belonging to the interior of a face $F \subseteq M$ of codimension k, has a neighborhood $V_p \cong V_p' \times [0, \varepsilon)^k$, with the following two properties: the defining function x_j is obtained as the projection onto the jth component of $[0, \varepsilon)^k$ and the metric h can be written as $h = h_F + (dx_1)^2 + \ldots (dx_k)^2$, with $x_1, \ldots , x_k$ being

the defining functions of F and h_F being a two-tensor that does not depend on $x_1, \dots, x_k$ and restricts to a metric on F.

Then, we consider on M the vector fields X that in a neighborhood of each point p, as above, are of the form

$$X = X_F + \sum_{j=1}^{k} x_j^{c_j} \partial_{x_j},$$

with c_j being the integer associated to the hyperface $\{x_j = 0\}$ and X_F being the lift of a vector field on F. The set of all vector fields with these properties forms a Lie subalgebra of the algebra of all vector fields on M. We denote this subalgebra by $\mathcal{A}(M, c)$. By the Serre-Swan theorem, there exists a vector bundle $A(M, c)$ such that $\mathcal{A}(M, c)$ identifies with the space of smooth sections of $A(M, c)$.

We want to integrate $A(M, c)$, and to this end, we shall use the approach from [29]. Let $S = int(F)$ be the interior of a face $F \subset M$ of codimension k. The restriction of $A(M, c)$ to each open face S is then $TS \times \mathbb{R}^k$, and hence it is integrable; a groupoid integrating this restriction being, for example $\mathcal{G}_S = S \times S \times \mathbb{R}^k$, if $F = \overline{S}$ has codimension k.

Define then

$$\mathcal{G} := \bigcup_F S \times S \times \mathbb{R}^k,$$

which is a groupoid with the obvious induced structural maps. As a set, $\mathcal{G}$ does not depend on c. Because the groupoids $\mathcal{G}_S$ are not d-connected, in general, we cannot use the result of [29] to prove that it has a natural smooth structure, so we have to construct this smooth structure directly.

The results of [29] say that if there exists a smooth structure on $\mathcal{G}$ compatible with its groupoid structure, then it must be obtained using certain coordinate charts defined using the exponential map. In our case, the exponential map amounts to the following.

Let $\psi_l : (0, \infty) \to \mathbb{R}$ be $\psi_l(x) = \ln x$, if $l = 1$ and $\psi_l(x) = x + x^{1-l}/(1-l)$, if $l > 1$. Also, let $\phi_l : \mathbb{R} \times [0, \infty) \to [0, \infty)$ be defined by $\phi_l(t, 0) = 0$ and $\phi_l(t, x) = \psi_l^{-1}(\psi_l(x) + t)$. In particular, $\phi_1(t, x) = e^t x$. Then ϕ_l defines a differentiable action of $\mathbb{R}$ on $[0, \infty)$, which hence makes $\mathbb{R} \times [0, \infty)$ a differentiable groupoid denoted $\mathcal{F}_l$. The Lie algebroid of $\mathcal{F}_l$ is generated as a $\mathcal{C}^\infty([0, \infty))$-projective module by the infinitesimal generator ∂_t of the action of $\mathbb{R}$; note that the action of ∂_t on $\mathcal{C}^\infty([0, \infty))$ under the anchor map is given by $f(x) x^l \partial_x$ for some nowhere vanishing (bounded) smooth function f. Consequently, $A(\mathcal{F}_l)$ is the projective $\mathcal{C}^\infty([0, \infty))$-module generated by $x^l \partial_x$.

Assume now that $M = [0, \infty)$ and fix $l \in \mathbb{N}$. Then $\mathcal{F}_l$ is a smooth groupoid integrating $A(M, l)$, by the above remarks. Consequently, if $M = [0, \infty)^n$ and $c = (c_1, c_2, \dots, c_n)$, then $\mathcal{G} := \mathcal{F}_{c_1} \times \mathcal{F}_{c_2} \times \dots \times \mathcal{F}_{c_n}$ satisfies $A(\mathcal{G}) = A(M, c)$. To integrate general Lie algebroids of the form $A(M, c)$ we localize this construction. This then gives the following smooth structure on $\mathcal{G} := \bigcup_S \mathcal{G}_S$.

We now discuss the general case of a manifold with corners. Locally, the smooth structure on $\mathcal{G}$ is given by the discussion above. Since this smooth structure is important in applications, let us try to make it more explicit. Thus, fix an arbitrary point $(p, q, \xi) \in S \times S \times \mathbb{R}^k$, which we want to include in a coordinate system. By definition, $p, q \in S$. Choose now a small coordinate neighborhood $V_p \cong V_p' \times [0, \varepsilon)^k$ of p, with V_p' a small open neighborhood of $p \in S$, as above. Choose $V_q \cong V_q' \times [0, \varepsilon)^k$

similarly. We write

$$z = (z', x_1(z), x_2(z), \dots, x_k(z))$$

for any $z \in V_p \cap V_q$; this is possible since we can assume that $V_p = V_q$ if $p = q$ or that $V_p \cap V_q = \emptyset$ if $p \neq q$. Fix $R > 2\|\xi\|$ and choose $\delta > 0$ so small that $|\phi_l(t, x)| < \varepsilon$ if $|t| \leq R$, $x \leq \delta$, and $l = c_j$, for $j = 1, 2, \dots, k$. Here c_j is the constant associated to the hyperface $\{x_j = 0\}$. Then we define a map

$$\begin{aligned} F : V_p' \times [0, \delta)^k \times V_q' \times \{\|\xi\| < R\} \quad &\longrightarrow \quad \mathcal{G} : \\ (z', y, z'', \xi) \quad &\longmapsto \quad (z', y, z'', \Phi(\xi, y), p_y(\xi)) \in \mathcal{G}_{S'} \end{aligned} \tag{19}$$

as follows. Let $y = (y_1, \dots, y_k)$, $B \subset \{1, 2, \dots, k\}$ be the subset of those indices j such that $y_j = 0$, and let $p_y : \mathbb{R}^k \to \mathbb{R}^B$ be the corresponding projection. The vector space $\mathbb{R}^B$ identifies naturally with the fiber at (z', y) of the normal bundle to the open face containing (z', y) (this open face was denoted above by $S' = S'(z', y)$). For $y = (y_1, y_2, \dots, y_k)$ and $\xi = (\xi_1, \xi_2, \dots, \xi_k)$, the map Φ is then given by

$$\Phi(\xi, y) = (\phi_{c_1}(\xi_1, y_1), \phi_{c_2}(\xi_2, y_2), \dots, \phi_{c_k}(\xi_k, y_k)) \,.$$

We shall denote by $\mathcal{G}(M, c)$ the smooth groupoid constructed above.

Fix a face $F \subset M$ of codimension k. By construction, F is an invariant subset of M and hence we can consider the restriction maps $\mathcal{R}_F$ defined in Equation (11). The range of these restriction (or indicial) maps is in related to the algebras $\Psi^\infty(\mathcal{G}(M, c))$. The precise relation is the following.

Each hyperface H' of F is a connected component of $H \cap F$, for a unique hyperface H of M. Then we associate to H' the integer $c_H \geq 1$. We denote by c' the collection of integers obtained in this way. Then the restriction of $A(M, c)$ to F is isomorphic to $A(F, c') \times \mathbb{R}^k$. From this we obtain that $\mathcal{G}(M, c)|_F \cong \mathcal{G}(F, c') \times \mathbb{R}^k$. The restriction maps thus become

$$\mathcal{R}_F : \Psi^m(\mathcal{G}(M, c)) \to \Psi^m(\mathcal{G}(F, c') \times \mathbb{R}^k).$$

The right hand side algebras are closely related to the "k-fold suspended algebras" of Melrose.

The analytic properties of the algebras $\Psi^\infty(\mathcal{G}(M, c))$ will be studied again in Section 10.

Example 22. Let M be a compact manifold whose boundary ∂M is the total space of a locally trivial fibration $p : \partial M \longrightarrow B$ of compact smooth manifolds. A smooth vector field on M is called an *edge vector field* if it is tangent to the fibers of p at the boundary. The Lie algebra $\mathcal{V}_e(M)$ of all edge vector fields is a projective $\mathcal{C}^\infty(M)$-module, and hence, by the Serre-Swan theorem [9], it can be identified with the space of all $\mathcal{C}^\infty$ sections of a smooth vector bundle ${}^eTM \to M$ that comes equipped with a natural map ${}^eTM \longrightarrow TM$ [19] making $A := {}^eTM$ into a Lie algebroid. A pseudodifferential calculus adapted to this setting was constructed by Mazzeo [19], and, in a slightly different way by Schulze [43]. To integrate ${}^eTM$, we shall use the methods of [29].

Let $M_0 := M \smallsetminus \partial M$ and notice that $A|_{M_0} \cong TM_0$. We can integrate this restriction to the pair groupoid: $\mathcal{G}_{M_0} := M_0 \times M_0$. The restriction of A to the boundary is the crossed product of another Lie algebroid with $\mathbb{R}$:

$$A|_{\partial M} \cong (T_{vert}\partial M \times TB) \rtimes \mathbb{R}.$$

It is worthwhile do describe this restriction more precisely. As a vector bundle, A is the direct sum of three vector bundles: $T_{vert}\partial M$ (the vertical tangent bundle to the fibers of $\partial M \to B$), $p^*(TB)$ (the pull-back of the tangent bundle of B), and a trivial, one-dimensional real vector bundle. Thus, every section of A can be represented as a triple (X, Y, f), where X is a vector field on ∂M, tangent to the fibers of $\partial M \to B$, Y is a section of $p^*(TB)$, which is convenient to be represented as a section of the quotient $T\partial M/T_{vert}\partial M$, and $f \in \mathcal{C}^\infty(\partial M)$. Let ∇ be the Bott connection on $p^*(TB)$. The Lie algebra structure on $\Gamma(A)$ is then

$$[(X, Y, f), (X_1, Y_1, f_1)] = ([X, X_1], \nabla_X(Y_1) + fY_1 - \nabla_{X_1}(Y) - f_1Y, 0).$$

Let $G \to B$ be the bundle of Lie groups obtained as the cross-product of the bundle of commutative Lie groups TB with $\mathbb{R}$, the action of $t \in \mathbb{R}$ being as multiplication with e^t. The Lie algebroid (or the bundle of Lie groups associated to this bundle of Lie groups) is $A(G) = TB \oplus \mathbb{R}$, with the bracked defined as above: $[(Y, f), (Y_1, f_1)] = (fY_1 - f_1Y, 0)$. Then we can write $A|_{\partial M} = T_{vert}\partial M \times_B A(G)$. This writing immediately leads to a groupoid integrating $A|_{\partial M}$, namely the fibered product of a groupoid integrating $T_{vert}\partial M$ and a groupoid integrating $A(G)$. We can choose these groupoids to be the fibered pair groupoid $\mathcal{G}_1 := \partial M \times_B \partial M$ and, respectively, G. The resulting groupoid integrating $A|_{\partial M}$ is then $\mathcal{G}_{\partial M} := \mathcal{G}_1 \times_B G \to B$, invariant with respect to the action of G by right translations. The resulting algebra of pseudodifferential operators will be an algebra of smooth families acting on the fibers of $\partial M \times_B G$, invariant with respect to G.

To obtain a groupoid integrating A, it is enough to show that the disjoint union $\mathcal{G} := \mathcal{G}_1 \cup (M_0 \times M_0)$ has a smooth structure compatible with the groupoid structure. This smooth structure is obtained using the following coordinate charts. Let x be a boundary defining function on M, and fix $q \in \partial M$ and a neighborhood $V_q \cong V_q' \times [0, \varepsilon)$ such that the defining functions x becomes the second projection on V_q and V_q' is a neighborhood of q in ∂M. We replace V_q with a smaller neighborhood, if necessary, so that there exists a fiber preserving diffeomorphism $\phi : B_1 \times B_2 \to V_q'$, $\phi(0, 0) = q$, from a product of two small open balls in some Euclidean spaces (so p becomes the first projection with respect to the diffeomorphism ϕ). Let $q' \in \partial M$ be a second point, chosen such that $p(q') = p(q)$, and choose a diffeomorphism $\phi' : B_1 \times B_2' \to V_{q'}$ as above. We can assume that $p \circ \phi = p \circ \phi'$ and $\phi = \phi'$, if $q = q'$, or that V_q and $V_{q'}$ are disjoint. Then ϕ and ϕ' define a diffeomorphism

$$\phi \times_B \phi' : B_1 \times B_2 \times B_2' \to \partial M \times_B \partial M,$$

explicitly,

$$\phi \times_B \phi'(b_1, b_2, b_2') = (\phi(b_1, b_2), \phi'(b_1, b_2')) \in \partial M \times_B \partial M \subset \partial M \times \partial M.$$

We identify B_1 with the fiber of TB at $p(q) = p(q')$ such that $p(q)$ corresponds to 0, and we let

$$\Phi : B_1 \times B_2 \times [0, \delta) \times B_1 \times B_2 \times (-R, R) \to \mathcal{G}$$

be given by

$$\Phi(b_1, b_2, 0, b_1', b_2', t) = \big(\phi \times_B \phi'(b_1, b_2, b_2'), b_1', t) \in \partial M \times_B \partial M \times T_{p(q)}B \times \mathbb{R} \in \mathcal{G}_1$$

or by

$$\begin{aligned} \Phi(b_1, b_2, s, b_1', b_2', t) &= (\phi(b_1, b_2), s, \phi'(b_1 + sb_1', b_2'), se^t) \\ &\in V_q' \times (0, \varepsilon) \times V_{q'}' \times (0, \varepsilon) \subset M_0 \times M_0. \end{aligned}$$

The restriction at the boundary map $\mathcal{R}_{\partial M}$ defined in Equation (11) becomes a map

$$\mathcal{R}_{\partial M} : \Psi^\infty(\mathcal{G}) \to \Psi^\infty(\mathcal{G}_1).$$

The range of this map consists of families of pseudodifferential operators that act on the fibers of $\partial M \times_B G \to B$ and are G-invariant with respect to the action of G by right translations.

We expect that the above example will be useful for the question from [5] on the Bojarsky additivity formula for the real index of families of elliptic operators. (See also Nicolaescu's paper [27].) Also, it will probably be useful for a certain approach in the study of the S^1-equivariant Dirac operators [31] and [16].

6. Geometric operators

For two vector bundles E_0, E_1 on M, we shall denote by $\mathrm{Diff}(\mathcal{G}; E_0, E_1)$ the space of differential operators $D : \Gamma(\mathcal{G}; r^*E_0) \to \Gamma(\mathcal{G}; r^*E_1)$ with smooth coefficients that differentiate only along the fibers of $d : \mathcal{G} \to M$, and which are right invariant. Thus, $\mathrm{Diff}(\mathcal{G}; E_0, E_1)$ is exactly the space of differential operators in $\Psi^m(\mathcal{G}; E_0, E_1)$. The elements of $\mathrm{Diff}(\mathcal{G}; E_0, E_1)$ will be called *differential operators on $\mathcal{G}$*.

In this section, we define and study the geometric differential operators on a given differentiable groupoid $\mathcal{G}$. For the definition of most of these operators, we shall need a metric on $A := A(\mathcal{G})$.

To define the de Rham operator, however, we need no metric. Denote then by $\lambda_q = \Lambda^q T^*_{vert}\mathcal{G}$ the qth exterior power of the dual of $T_{vert}\mathcal{G}$, the vertical tangent bundle to the fibers of $d : \mathcal{G} \to M$. Recall that $\mathcal{G}_x$ denotes $d^{-1}(x)$ throughout this paper. Then the de Rham differential $d : \Gamma(\mathcal{G}_x, \lambda_q) \to \Gamma(\mathcal{G}_x, \lambda_{q+1})$ is invariant with respect to right translations, and hence it defines an operator $d \in \bigoplus_q \mathrm{Diff}(\mathcal{G}; \Lambda^q A^*, \Lambda^{q+1} A^*) \subset \mathrm{Diff}(\mathcal{G}; \Lambda^* A^*)$.

Let as before π be the representation $\pi : \Psi^m(\mathcal{G}; E_0, E_1) \to \mathrm{Hom}(\Gamma(E_0), \Gamma(E_1))$ given by the formula $\pi(P)(f) \circ r = P(f \circ r)$. (Recall that we called this representation the *vector representation*.) The complex determined by the operators $\pi(d)$ then computes the *Lie algebroid cohomology* of A, by definition, see [17].

We shall use later on the explicit form of d for the adiabatic groupoid associated to $\mathcal{G}$. Recall that an operator P (differential or pseudodifferential) on the adiabatic groupoid $\mathcal{G}_{\mathrm{ad}}$ associated to $\mathcal{G}$ consists of a family $P = (P_t)$, such that, in particular, $P_t \in \Psi^m(\mathcal{G})$, for $t > 0$. The de Rham operator $d^{\mathcal{G}_{\mathrm{ad}}} = (d_t)$ is such that $d_t = td$, for $t > 0$.

Of course, more operators are obtained if we consider a metric on $A := A(\mathcal{G})$. Here, by "metric on A" we mean a positive definite bilinear form on A, as usual. The metric on A then makes each $\mathcal{G}_x$ a Riemannian manifold, naturally, due to the isomorphisms $T\mathcal{G}_x \cong r^*(A)$ as vector bundles on $\mathcal{G}_x$. Moreover, right translation by an element of $\mathcal{G}$ is an isometric isomorphism. Because of this, every geometric differential operator associated naturally to a Riemannian metric will be invariant with respect to the right translation by an element of $\mathcal{G}$, and hence will define an element in $\mathrm{Diff}(\mathcal{G}; E_0, E_1)$, for suitable vector bundles E_i. We shall not try to formulate this in the greatest generality, but we shall apply this observation to particular operators that appear more often in practice.

For example, the metric allows us to define the Hodge $*$-operator, which then leads to the signature operator $d \pm *d* \in \mathrm{Diff}(\mathcal{G}, \Lambda^* A)$. Also, the metric gives rise to an inner product on $\Lambda^* A$ and hence to an adjoint to d, denoted d^*, which then

in turn allows us to define the Euler operator $d+d^* \in \mathrm{Diff}(\mathcal{G};\Lambda^* A^*)$. Similarly, one obtains the Hodge Laplacians $\Delta_p \in \mathrm{Diff}(\mathcal{G};\Lambda^p A^*)$, as components of the square of the Euler operator $d+d^*$. We write $\Delta_p^{\mathcal{G}}$, $d^{\mathcal{G}}$, ... for these operators when we want to stress their dependence on $\mathcal{G}$.

We now turn to Dirac and generalized Dirac operators. This requires us to introduce the (generalization to groupoids of the) Levi-Civita connection.

For $X \in \Gamma(A)$, we shall denote by $\tilde{X}$ its lift to a right invariant, d-vertical vector field on $\mathcal{G}$. Let $\nabla^x : \Gamma(T_{vert}\mathcal{G}_x) \to \Gamma(T_{vert}\mathcal{G}_x \otimes T^*_{vert}\mathcal{G}_x)$ be the Levi-Civita connection associated to the induced metric on $\mathcal{G}_x$. Then for any $X \in \Gamma(A)$, we obtain a smooth, right invariant family of differential operators

$$\nabla^x_{\tilde{X}} : \Gamma(T_{vert}\mathcal{G}_x) \to \Gamma(T_{vert}\mathcal{G}_x).$$

We denote the induced differential operator in $\mathrm{Diff}(\mathcal{G}, A)$ simply by ∇_X. For all smooth sections X and Y of A, there exists another smooth section Z of A such that $\nabla_X(\tilde{Y}) = \tilde{Z}$.

Suppose now that A is *spin*, that is, that A is orientable and the bundle of orientable frames of A lifts to a principal $Spin(k)$ bundle (k being the rank of A). Suppose $k = 2l$ is even, for simplicity, and let $S = S_+ \oplus S_-$ be the spin bundle associated to the given spin structure and the spin representation of $Spin(k)$. As in the classical case, the Levi-Civita connection on the frame bundle of $r^*(A)$ lifts to a connection ∇^S on S. Moreover, this connection involves no choices (it is uniquely determined by the spin structure), and hence ∇^S is right invariant, in the obvious sense. Thus, if X is a section of A and $\tilde{X}$ is its lift to a right invariant, d-vertical vector field on $\mathcal{G}$, then $\nabla^S_{\tilde{X}}$ is a right invariant differential operator, and hence it is in $\mathrm{Diff}(\mathcal{G}, S)$. We denote by $\not{D}^S$ the induced Dirac operator on the spaces $\mathcal{G}_x$, which will then form a right invariant family, and hence $\not{D}^S \in \mathrm{Diff}(\mathcal{G}; S)$. (We shall write $\not{D}^S_{\mathcal{G}}$ on the few occasions when we shall need to point out the dependence of this operator on the groupoid $\mathcal{G}$.)

Let $\mathrm{Cliff}(A)$ be the bundle of Clifford algebras associated to A and its metric. We shall use the metric to identify A^* with A, so that $\mathrm{Cliff}(A^*)$ becomes identified with $\mathrm{Cliff}(A)$. The same construction as above then applies to a $\mathrm{Cliff}(A)$-module W endowed with a right invariant, admissible connection ∇^W (see below) on each of its restrictions to $\mathcal{G}_x$. Denote by $c : \mathrm{Cliff}(A) \to End(W)$ the Clifford module structure on W. Because $A \subset \mathrm{Cliff}(A)$, we also obtain a bundle morphism $A \to End(W)$ still denoted c. Recall then that ∇^W is an *admissible connection* if, and only if,

$$\nabla^W_X(c(Y)\xi) = c(\nabla_X Y)\xi + c(Y)\nabla^W_X(\xi),$$

for all $\xi \in \Gamma(r^*(W))$ and all $X, Y \in \Gamma(r^*(A))$, the second connection being the Levi-Civita connection discussed above. Then we obtain as in the classical case a Dirac operator $\not{D}^W_x$ on $\mathcal{G}_x$, acting on sections of $r^*(W)$. The right invariance of the connection ∇^W guarantees that the family $\not{D}^W_x$ is right invariant, and hence that it defines an element in $\mathrm{Diff}(\mathcal{G}; W)$.

It is a little bit trickier to define the generalized Dirac operator associated to a $\mathrm{Cliff}(A)$-module W, if no admissible connection is specified on W. This is because it is not clear a priori that right invariant admissible connections exist at all. Our next goal then is to prove that this is always the case, as it is for Clifford modules on Riemannian manifolds.

We shall work with complex $\mathrm{Cliff}(A)$-modules, for simplicity. Also, we assume that A is even dimensional. Cover M with contractible open sets U_α. Then $A|_{U_\alpha}$

has a trivialization $A|_{U_\alpha} \simeq U_\alpha \times \mathbb{R}^{2l}$, which we can assume to preserve the metric. Then $\mathrm{Cliff}(A)|_{U_\alpha} \simeq U_\alpha \times M_{2^l}(\mathbb{C})$ and $W|_{U_\alpha} \simeq U_\alpha \times V \simeq \mathbb{C}^{2^l} \otimes V_0$, with V_0 an additional vector bundle, which is acted upon trivially by the Clifford algebra, and hence only serves to encode the local "multiplicity" of the $\mathrm{Cliff}(A)$-module W. As in the classical case, we first define the admissible connection locally, using the above trivialization, and then we glue them using a partition of unity. However, in our groupoid setting we need to work a little bit more to make sense of what the "local definition" means. More precisely, all definitions will be given not on U_α itself, but on $r^{-1}(U_\alpha)$. Once we realize this, everything carries over from the case of a Riemannian manifold to that of a differentiable groupoid. For completeness, we now review this construction in our case.

The trivialization of U_α gives an orthonormal family of sections $X_1, X_2, \dots, X_k$ of A over U_α. Then, we obtain smooth functions Γ^a_{bc} on U_α such that, working *always* over $r^{-1}(U_\alpha)$,

$$\nabla_{\tilde{X}_i} \tilde{X}_j = \sum_h (\Gamma^h_{ij} \circ r) \tilde{X}_h.$$

(Compare with [15].) Fix a basis (e_t), $t = 1, \dots, 2^l m$, of V, where V is the vector space appearing in the isomorphism $W|_{U_\alpha} \simeq U_\alpha \times V$. We shall denote by $\tilde{e}_t := e_t \circ r$ the induced basis of $r^*(W)$ on $r^{-1}(U_\alpha)$. The point of these choices is, of course, that the matrix of the multiplication operator $c(\tilde{X}_j)$ in the basis $\tilde{e}_t$ consists of *constant* functions. Using the functions Γ^h_{ij} and the Clifford multiplication map $c : A \to End(W)$, we define a connection $\nabla^{x,W,\alpha}$ on the restriction of $r^*(W)$ to $\mathcal{G}_x \cap r^{-1}(U_\alpha)$ by the formula

$$\nabla^{x,W,\alpha}_{\tilde{X}_h} \tilde{e}_t := \frac{1}{4} \sum_{a,b} (\Gamma^b_{ha} \circ r) c(\tilde{X}_a) c(\tilde{X}_b) \tilde{e}_t. \tag{20}$$

Let $\phi_\alpha \in \mathcal{C}^\infty(M)$ be a $\mathcal{C}^\infty$-partition of unity subordinate to the covering U_α. Then $\tilde{\phi}_\alpha := \phi_\alpha \circ r$ is a partition of unity subordinate to $r^{-1}(U_\alpha)$. We define a connection $\nabla^{x,W}$ on the restriction of W to $\mathcal{G}_x$ by the formula

$$\nabla^{x,W}_{\tilde{X}}(\xi) = \sum_\alpha \nabla^{x,W,\alpha}(\tilde{X})(\tilde{\phi}_\alpha \xi).$$

By the definition, $\nabla^{x,W}$ is an admissible connection on the restriction of $r^*(W)$ to $\mathcal{G}_x$.

Proposition 3. *Let $W \to M$ be a complex vector bundle that is a $\mathrm{Cliff}(A)$-module. Then we can find an admissible connection $\nabla^{x,W}$ on the restriction of $r^*(W)$ to $\mathcal{G}_x$, for any $x \in M$, such that for each $X \in \Gamma(A)$, the operators $\nabla^{x,W}_{\tilde{X}}$ form a smooth, $\mathcal{G}$-invariant family of differential operators on $r^*(W)$, and hence they define an element ∇^W_X in $\mathrm{Diff}(\mathcal{G}; W)$. If S is a spin bundle, then we can take this connection to be the Levi-Civita connection.*

Proof. This is just the summary of the above discussion. ☐

It follows from the above proposition that if we consider on each $\mathcal{G}_x$ the Dirac operator determined by the connection $\nabla^{x,W}$, then we obtain an invariant family of differential operators, which hence defines an operator $\not{D}^W_{\mathcal{G}} \in \mathrm{Diff}(\mathcal{G}; W)$, the Dirac operators on $\mathcal{G}$ associated to W and the given admissible connection. (When the groupoid $\mathcal{G}$ is clear from the context, we shall drop the subscript $\mathcal{G}$.)

We can also regard the admissible connection on a Cliff(A)-module W as an operator $\nabla^W \in \mathrm{Diff}(\mathcal{G}; W, W \otimes A^*)$. If we denote by $c \in Hom(W \otimes A^*, W)$ the Clifford multiplication, then, as in the classical case $\not{D}^W = c \circ \nabla^W$. We can also generalize the local description of Dirac operators. Let $M = \bigcup U_\alpha$ be a covering of M by open subsets which trivializes the bundle $A = A(\mathcal{G})$, and choose a partition of unity ϕ_α^2 subordinate to U_α. On each U_α, we choose a local *orthonormal* basis $X_1, \dots, X_k$ of A and define $X_j^\alpha = \phi_\alpha X_j$. Then

$$\not{D}^W = \sum_{\alpha, j} c(X_j^\alpha) \nabla^W_{X_j^\alpha}. \tag{21}$$

As in the classical case of a Riemannian manifold, the space of $\mathcal{G}$-invariant, admissible connections $\nabla^{x,W}$ on $r^*(W)$ is an affine space with model vector space the space of skew-adjoint elements in the space of Cliff(A)-linear endomorphisms of W.

A feature specific to the groupoid case, however, is that all the above constructions and operators are compatible with restrictions to compact, $\mathcal{G}$-invariant subsets of M. (Recall that a subset $Y \subset M$ of the space of units of $\mathcal{G}$ is $\mathcal{G}$-invariant if, and only if, $d^{-1}(Y) = r^{-1}(Y)$.) For instance, consider a Cliff(A) bundle W on M with admissible connection ∇. Then W restricts to a Cliff$(A|_Y)$ module on Y. From this observation we get that the Dirac operator on $\mathcal{G}$ associated to the Cliff(A)-module W will restrict to the Dirac operator on $\mathcal{G}_Y := d^{-1}(Y)$ associated to the Cliff$(A|_Y)$-module $W|_Y$. Formally,

$$\mathcal{R}_Y(\not{D}^W_{\mathcal{G}}) = \not{D}^{W_Y}_{\mathcal{G}_Y}. \tag{22}$$

Similarly,

$$\mathcal{R}_Y(\Delta_p^{\mathcal{G}}) = \Delta_p^{\mathcal{G}_Y}, \quad \mathcal{R}_Y(d^{\mathcal{G}}) = d^{\mathcal{G}_Y}, \tag{23}$$

and so on. This leads, as we shall see in the following sections, to Fredholmness criteria for these various operators in terms of the invertibility of the corresponding operators associated to proper, invariant, closed submanifolds.

7. Sobolev spaces

Throughout this section, we assume for simplicity, that the space M of units of a given groupoid $\mathcal{G}$ is compact. All the definitions and results extend to the case of sections of a Hermitian vector bundle E and operators acting on sections of E. For simplicity, however, we shall discuss in detail only the case where E is the one-dimensional, trivial bundle.

The notation E, sometimes is used in this section to denote the identity element of operator algebras, in this section.

Consider a bounded, non-degenerate representation $\varrho : \Psi^{-\infty}(\mathcal{G}) \longrightarrow \mathrm{End}(\mathcal{H})$. Theorem 2 then gives a natural extension of ϱ to a bounded $*$-representation of $\Psi^\infty(\mathcal{G})$. Here "bounded" refers to the fact that the order zero operators act by bounded operators on $\mathcal{H}$, see Definition 8. Recall that ϱ is non-degenerate if the space $\mathcal{H}_\infty := \varrho(\Psi^{-\infty}(\mathcal{G}))\mathcal{H}$ is dense in $\mathcal{H}$. The best we can hope for formally self-adjoint operators $A = A^* \in \Psi^m(\mathcal{G})$ is that they are essentially self-adjoint unbounded operators on $\mathcal{H}$. This is in fact the case for elliptic operators; for $m > 0$, a formally self-adjoint, elliptic operator $A = A^* \in \Psi^m(\mathcal{G})$ leads to densely defined, essentially self-adjoint operator

$$\varrho(A) : \mathcal{H}_\infty \longrightarrow \mathcal{H}.$$

This will allow us to freely use functional calculus for self-adjoint operators later on in this section. Fix a bounded, non-degenerate representation ϱ as above.

Note that under the assumptions above, the unit $E := (\mathrm{id}_{\mathcal{G}_x})_{x\in M}$ belongs to $\Psi^0(\mathcal{G})$ with $\varrho(E) = \mathrm{id}_{\mathcal{H}}$. Let further $\mathcal{H}_{-\infty} := \mathcal{H}_\infty^*$ be the algebraic dual of $\mathcal{H}_\infty$, and $T : \mathcal{H} \hookrightarrow \mathcal{H}_{-\infty} : h \mapsto T_h$ be the natural, anti-linear embedding. As in the classical case, ϱ induces a multiplicative morphism $\widetilde{\varrho} : \Psi^\infty(\mathcal{G}) \longrightarrow \mathrm{End}(\mathcal{H}_{-\infty})$ by

$$[\widetilde{\varrho}(A)u](\xi) := u(\varrho(A^*)\xi)$$

for $A \in \Psi^\infty(\mathcal{G})$, $u \in \mathcal{H}_{-\infty}$, and $\xi \in \mathcal{H}_\infty$. Chasing definitions yields

$$\begin{aligned} &\widetilde{\varrho}(A)\circ T = T\circ\varrho(A) \quad : \quad \mathcal{H}_\infty \longrightarrow \mathcal{H}_{-\infty} \text{ for } A \in \Psi^\infty(\mathcal{G})\,, \text{ and} \\ &\widetilde{\varrho}(A)\circ T = T\circ\varrho(A) \quad : \quad \mathcal{H} \longrightarrow \mathcal{H}_{-\infty} \text{ for } A \in \Psi^0(\mathcal{G})\,. \end{aligned} \tag{24}$$

Recall that an unbounded, closable operator $S : \mathcal{H} \supseteq \mathcal{D}(S) \longrightarrow \mathcal{H}$ is called *essentially self-adjoint* if $\mathcal{D}(S)$ is dense in $\mathcal{H}$, and $\overline{S} = S^*$ where $\overline{S}$ denotes the minimal closed extension of S, and S^* its adjoint in the sense of unbounded operators. The proof of the following proposition is appropriately adapted from [45, Theorem 26.2].

Proposition 4. *Let $m > 0$, and $A = A^* \in \Psi^m(\mathcal{G})$ be elliptic. Then the unbounded operator $\varrho(A) : \mathcal{H} \supseteq \mathcal{H}_\infty \longrightarrow \mathcal{H}$ is essentially self-adjoint. Moreover,*

$$\mathcal{D}\left(\overline{\varrho(A)}\right) = \mathcal{D}(\varrho(A)^*) = \{h \in \mathcal{H} : \widetilde{\varrho}(A)T_h \in T\mathcal{H}\}\ . \tag{25}$$

Proof. For brevity, let $\mathcal{D}$ be the space on the right-hand side in (25). Also, let $h \in \mathcal{D}(\varrho(A)^*)$. Then we get for all $\xi \in \mathcal{H}_\infty$

$$\widetilde{\varrho}(A)T_h(\xi) = (\varrho(A)\xi, h) = (\xi, \varrho(A)^*h) = T_{\varrho(A)^*h}(\xi)\,,$$

i.e. $\mathcal{D}(\varrho(A)^*) \subseteq \mathcal{D}$, and $\widetilde{\varrho}(A)T_h = T_{\varrho(A)^*h}$.

On the other hand, for $h \in \mathcal{D}$, there exists $g \in \mathcal{H}$ such that for all $\xi \in \mathcal{H}_\infty$

$$(\varrho(A)\xi, h) = \widetilde{\varrho}(A)T_h(\xi) = T_g(\xi) = (\xi, g)\,,$$

hence, $h \in \mathcal{D}(\varrho(A)^*)$ which gives the second equality in (25). By [45, Theorem 26.1], it remains to show

$$N(\varrho(A)^* \pm i\,\mathrm{id}_{\mathcal{H}}) \subseteq \mathcal{D}\left(\overline{\varrho(A)}\right)\ .$$

Because of $m > 0$, $A \pm iE \in \Psi^m(\mathcal{G})$ is elliptic; by the usual symbolic argument we get $B_\pm \in \Psi^{-m}(\mathcal{G})$ satisfying $E - B_\pm(A \pm iE) =: R_\pm \in \Psi^{-\infty}(\mathcal{G})$. Furthermore, for $\xi \in N(\varrho(A)^* \pm i\,\mathrm{id}_{\mathcal{H}}) \subseteq \mathcal{D}(\varrho(A)^*)$ another definition chase yields as before

$$\widetilde{\varrho}(A \pm iE)T_\xi = T((\varrho(A)^* \mp i\,\mathrm{id}_{\mathcal{H}})\xi) = 0\,,$$

thus,

$$T_\xi = \widetilde{\varrho}(R_\pm)T_\xi = T_{\varrho(R_\pm)\xi} \in T\mathcal{H}_\infty$$

because of $R_\pm \in \Psi^{-\infty}(\mathcal{G})$ and (24). Since we have $\mathcal{H}_\infty \subseteq \mathcal{D}\left(\overline{\varrho(A)}\right)$, this completes the proof. □

Let us now define Sobolev spaces in the setting of groupoids using the powers of an arbitrary positive element $D \in \Psi^m(\mathcal{G})$, $m > 0$, as customary. The necessary facts that imply independence of D are contained in the following theorem (and the lemmata leading to its proof). Also, the following theorem will allow us to reduce certain questions about operators of positive order to operators of order zero.

We shall write $P \geq 0$ if $P = P^* \in \Psi^m(\mathcal{G})$ is such that $(\varrho(P)\xi, \xi) \geq 0$ for all $\xi \in \mathcal{H}_\infty$ and for every non-degenerate representation ϱ of $\Psi^\infty(\mathcal{G})$ on $\mathcal{H}$. Also, we

shall write $A \geq B$ if $A - B \geq 0$. For $Q \in \mathfrak{A}(\mathcal{G})$, we write $Q = P^{-1}$, if, and only if, $\varrho(P)\varrho(Q) = \mathrm{id}_{\mathcal{H}}$ and $\varrho(Q)\varrho(P) \subseteq \mathrm{id}_{\mathcal{H}}$, for every non-degenerate, bounded representation ϱ. Then, for $s > 0$, P^{-s} stands for $(P^{-1})^s$.

Theorem 4. *Fix a differentiable groupoid $\mathcal{G}$ whose space of units, M, is compact. Let $D \in \Psi^m(\mathcal{G})$, $m > 0$, be such that $D \geq E$ and $\sigma_m(D) > 0$. Then $D^{-s} \in C^*(\mathcal{G})$, for all $s > 0$. Moreover, if P has order $\leq k$, then $PD^{-k/m} \in \mathfrak{A}(\mathcal{G})$.*

The proof will consist of a sequence of lemmata.

Lemma 2. *Fix arbitrarily a metric on $A(\mathcal{G})$, and let $B = E + \Delta$, where Δ is the positive Laplace operator on functions. Then B is invertible in the sense above, and we have $B^{-1} \in C^*(\mathcal{G})$.*

Proof. Let $D_t = E + t^2\Delta$, $t > 0$. We shall prove first that, for small t, there exists $Q_t \in C^*(\mathcal{G})$ such that $\varrho(Q_t)\varrho(D_t) \subseteq \mathrm{id}_{\mathcal{H}}$ and $\varrho(D_t)\varrho(Q_t) = \mathrm{id}_{\mathcal{H}}$, for all non-degenerate representations ϱ on $\mathcal{H}$.

Because the family (td), $t > 0$, extends to a first-order differential operator on $\mathcal{G}_{\mathrm{ad}}$, the adiabatic groupoid of $\mathcal{G}$, we obtain that $t^2\Delta = (td)^*(td)$ induces an element in $\Psi^2(\mathcal{G}_{\mathrm{ad}})$, which explains the choice of the power t^2.

To be precise, let $E \in \Psi^0(\mathcal{G})$, $E_{\mathrm{ad}} \in \Psi^0(\mathcal{G}_{\mathrm{ad}})$, and $E_0 \in \Psi^0(A(\mathcal{G}))$ be the identity elements. If e_t, $t \geq 0$, denotes the evaluation map as in Example 20 (so that, in particular $e_t : \Psi^\infty(\mathcal{G}_{\mathrm{ad}}) \to \Psi^\infty(\mathcal{G})$, $t > 0$), then we have $e_t(E_{\mathrm{ad}}) = E$ for $t > 0$, and $e_0(E_{\mathrm{ad}}) = E_0$. Thus, the family (D_t) leads to an element $D \in \Psi^2(\mathcal{G}_{\mathrm{ad}})$. Choose a quantization map q for $\mathcal{G}_{\mathrm{ad}}$ as in [32], and denote by $|\xi|$ the metric on $A^*(\mathcal{G})$, so that the principal symbol of Δ is $|\xi|^2$. Then the function $p(t,\xi) := (1 + |\xi|^2)^{-1}$ is an order two symbol on $A^*(\mathcal{G}_{\mathrm{ad}})$, see Example 20, and $F := q(p)D \in \Psi^0(\mathcal{G}_{\mathrm{ad}})$ satisfies $e_0(F) = E_0$. ¿From the results [10, 11, 36], we know that the function $t \mapsto \|e_t(F - E_{\mathrm{ad}})\|$ is continuous at 0 (in fact everywhere, but that is all that is needed), and hence $e_t(F)$ will be invertible in $\mathfrak{A}(\mathcal{G})$ for t small. We define then

$$Q_t := e_t(F)^{-1} e_t(q(p)) \in \mathfrak{A}(\mathcal{G})\Psi^{-2}(\mathcal{G}) \subseteq C^*(\mathcal{G}),$$

and a straight-forward computation gives $\varrho(Q_t)\varrho(D_t)\xi = \xi$, for $\xi \in \mathcal{H}_\infty$, a dense subspace of $\mathcal{H}$, and for $t > 0$ but small. Since $\varrho(D_t)$ is (essentially) self-adjoint, we obtain that $\varrho(Q_t)$ is the inverse of (the closure of) $\varrho(D_t)$. This means $Q_t = D_t^{-1}$, for $t > 0$ but small, according to our conventions.

Let now $h_t(y) = (1 + t^2 y)^{-1}$ and $\varepsilon > 0$ be arbitrary. Then there exists a continuous function $g_{\varepsilon,t} : [0,1] \longrightarrow [0,1]$ with $g(0) = 0$ such that $h_t = g_{\varepsilon,t} \circ h_\varepsilon$, and we obtain $D_t^{-1} = g_{\varepsilon,t}(D_\varepsilon^{-1}) \in C^*(\mathcal{G})$ by the composition property of the functional calculus for continuous functions, for ε small enough. Because of $B = D_1$ this completes the proof. □

Lemma 3. *Let $D \in \Psi^m(\mathcal{G})$ be elliptic with $\sigma_m(D) > 0$. Then, for each $A \in \Psi^m(\mathcal{G})$ and for each bounded representation ϱ of $\Psi^\infty(\mathcal{G})$ on $\mathcal{H}$, we can find $C_A \geq 0$ such that $\|\varrho(A)f\| \leq C_A(\|f\| + \|\varrho(D)f\|)$, for all $f \in \mathcal{H}_\infty := \varrho(\Psi^{-\infty}(\mathcal{G}))\mathcal{H}$.*

Proof. The proof is the same as that of the boundedness of operators of order zero, using Hörmander's trick [7]. Let us briefly recall the details.

It suffices to show $\|\varrho(A)f\|^2 \leq C(\|f\|^2 + \|\varrho(D)f\|^2)$, for some constant C independent of f. Choose $C_1 > 0$ with $|\sigma_m(A)|^2 \leq C_1|\sigma_m(D)|^2$. This is possible because $\sigma_m(A)\sigma_m(D)^{-1}$ is defined and continuous on the sphere bundle $S^*(\mathcal{G})$ of $A^*(\mathcal{G})$, a compact space. Let $b > 0$ be smooth with $b^2 = (C_1 + 1)|\sigma_m(D)|^2 - |\sigma_m(A)|^2$ (this

is defined only outside the zero section), and let $B \in \Psi^m(\mathcal{G})$ be an operator with principal symbol $\sigma_m(B) = b$. Then

$$(C_1 + 1)D^*D - A^*A - B^*B = R,$$

with R of order $l \leq 2m - 1$. By replacing B with B_1 such that $B_1 - B$ has order $l - m$ and $\sigma_{l-m}(B_1 - B) = \sigma_l(R)/2b$, we obtain that the order of the operator $(C_1 + 1)D^*D - A^*A - B_1^*B_1$ is less than l. Continuing in this way, we may assume that R has order ≤ 0, so in particular $\varrho(R)$ is bounded. Then

$$\|\varrho(A)f\|^2 \leq (C_1 + 1)(\varrho(D^*D)f, f) - (\varrho(R)f, f) \leq C(\|\varrho(D)f\|^2 + \|f\|^2)$$

for $C := \max\{\|\varrho(R)\|, C_1 + 1\}$. □

Lemma 4. *Let $D = D^* \in \Psi^m(\mathcal{G})$, be elliptic with $\sigma_m(D) > 0$. Then we can find $C \geq 0$ such that, for any bounded representation ϱ of $\Psi^\infty(\mathcal{G})$ on $\mathcal{H}$ we have $(\varrho(D)f, f) \geq -C(f, f)$, for all $f \in \mathcal{H}_\infty := \varrho(\Psi^{-\infty}(\mathcal{G}))\mathcal{H}$.*

Proof. The statement follows from the boundedness of D, if $m \leq 0$, so assume that $m > 0$. Then the proof is the same as that of the previous lemma if in the proof of that lemma we replace D^*D with D and take $A = 0$. □

For the rest of the proof of Theorem 4, we shall fix a non-degenerate representation ϱ of $\Psi^\infty(\mathcal{G})$ on $\mathcal{H}$, and we shall identify the elements of $\Psi^\infty(\mathcal{G})$ with unbounded operators with common domain $\mathcal{H}_\infty$, and the elements of $\mathfrak{A}(\mathcal{G})$ with bounded operators on $\mathcal{H}$.

Corollary 1. *If $D = D^* \in \Psi^m(\mathcal{G})$, $m > 0$, is such that $\sigma_m(D) > 0$, then there exists $C \geq 0$ such that $D + CE \geq E$. For any such $C \geq 0$ and any $A \in \Psi^m(\mathcal{G})$, the operator $A(D + C)^{-1}$ extends uniquely to a bounded operator on $\mathcal{H}$.*

Proof. The first statement follows from the previous lemma. Fix $C \geq 0$ such that $D + CE \geq E$. Lemma 3 gives $\|Af\| \leq C_1(\|f\| + \|(D + CE)f\|)$, for some $C_1 > 0$ and all $f \in \mathcal{H}_\infty$ Consequently, there is $C_2 > 0$ with

$$\|Af\| \leq C_2\|(D + C)f\|, \tag{26}$$

for all $f \in \mathcal{H}_\infty$. Since $D + CE$ is essentially self-adjoint by Proposition 4 and $D + CE \geq E$, its range $\mathcal{H}_1 := (D + CE)\mathcal{H}$ is dense by [37, Theorem X.26]. By (26), we obtain for $g = (D + CE)f \in \mathcal{H}_1$

$$\|A(D + CE)^{-1}g\| \leq C_2\|g\|\,,$$

which completes the proof. □

Corollary 2. *Consider now two self-adjoint, elliptic elements $D_1, D_2 \in \Psi^m(\mathcal{G})$, $m > 0$, with $D_i \geq E$ and $\sigma_m(D_i) > 0$, $i = 1, 2$. Then $D_1D_2^{-1}$ extends uniquely to a bounded invertible operator.*

Proof. By the previous corollary, both $D_1D_2^{-1}$ and $D_2D_1^{-1}$ extend to bounded operators. □

Lemma 5. *Let $D \in \Psi^m(\mathcal{G})$, $m > 0$, be such that $D \geq E$, $\sigma_m(D) > 0$, and $D^{-1} \in C^*(\mathcal{G})$. Then we have $PD^{-k}, D^{-k}P \in \mathfrak{A}(\mathcal{G})$, if P has order $\leq km$. Moreover, we have $\sigma_0(PD^{-k}) = \sigma_{km}(P)\sigma_m(D)^{-k}$, and $\sigma_0(D^{-k}P) = \sigma_m(D)^{-k}\sigma_{km}(P)$.*

Proof. We notice that if D satisfies the assumptions of the lemma, then D^k satisfies them as well. We can assume then that $k = 1$.

We shall check only that $PD^{-1} \in \mathfrak{A}(\mathcal{G})$. The relation $D^{-1}P \in \mathfrak{A}(\mathcal{G})$ can be proved in the same way or follows from the first one by taking adjoints.

Let $A \in \Psi^{-m}(\mathcal{G})$ be with $AD - E = R \in \Psi^{-\infty}(\mathcal{G})$, $B_n \in \Psi^{-\infty}(\mathcal{G})$ be a sequence converging to $D^{-1} \in C^*(\mathcal{G})$, and define $A_n := A - RB_n \in \Psi^{-m}(\mathcal{G})$. Then we have $A_n - D^{-1} = R(D^{-1} - B_n)$, thus, $PD^{-1} = PA_n - PR(D^{-1} - B_n)$ first defined on the dense subspace $D\mathcal{H}_\infty$, has a unique bounded extension with $PD^{-1} \in \mathfrak{A}(\mathcal{G})$ because of $PA_n \in \Psi^0(\mathcal{G})$ and

$$\|PA_n - PD^{-1}\| \leq \|PR\| \|D^{-1} - B_n\| \to 0 \,, \quad n \to \infty.$$

Since $\sigma_0(PD^{-1})$ is the limit of $\sigma_0(PA_n) = \sigma_m(P)\sigma_{-m}(A) = \sigma_m(P)\sigma_m(D)^{-1}$, we obtain the formula for the principal symbol as well. $\square$

Lemma 6. *Let $D \in \Psi^m(\mathcal{G})$ be with $D \geq E$ and $\sigma_m(D) > 0$. Then $D^{-1} \in C^*(\mathcal{G})$.*

Proof. From Lemma 2 we know that $(E + \Delta)^{-m}$ is in $C^*(\mathcal{G})$. By Corollary 2, applied to $D_1 = (E + \Delta)^m$ and $D_2 = D^2$, we have $(E + \Delta)^m D^{-2} = D_1 D_2^{-1} \in \mathfrak{A}(\mathcal{G})^{-1}$, thus, $D^{-2} = (E + \Delta)^{-m}(E + \Delta)^m D^{-2} \in C^*(\mathcal{G})$. Taking square roots completes the proof. $\square$

We now complete the proof of Theorem 4.

Proof. Assume first that $m = 1$. Then the theorem follows from Lemma 5 and Lemma 6. For arbitrary m, $D^{-1} \in C^*(\mathcal{G})$, and hence we get $D^{-s} \in C^*(\mathcal{G})$ by using functional calculus with continuous functions. A look at Lemma 5 completes the proof. $\square$

We now obtain some corollaries of Theorem 4.

For the following results, we need to define Sobolev spaces. Fix a metric on $A(\mathcal{G})$. Let then $\Delta := \Delta_0 \in \mathrm{Diff}(\mathcal{G})$ be the Hodge-Laplacian acting on functions, and ϱ be a non-degenerate, bounded representation as above. Then $D := \varrho(E + \Delta)$ is essentially self-adjoint and strictly positive, hence we can define D^s, for each $s \in \mathbb{R}$, using the functional calculus for essentially self-adjoint operators. Then $H^s(\mathcal{H}, \varrho)$, *the sth Sobolev space of $(\mathcal{H}, \varrho)$*, is by definition, the domain of $D^{s/2}$ with the graph topology, if $s \geq 0$, or its dual if $s < 0$.

Corollary 3. *The spaces $H^s(\mathcal{H}, \varrho)$ do not depend on the choice of the metric on $A(\mathcal{G})$, and every pseudodifferential operator $P \in \Psi^m(\mathcal{G})$ gives rise to a bounded map $H^s(\mathcal{H}, \varrho) \to H^{s-m}(\mathcal{H}, \varrho)$.*

Proof. If we change the metric on the compact space M, we obtain a new Laplace operator, and D will be replaced by a different operator D_1. However, by Corollary 2, $D^s D_1^{-s}$ and $D_1^s D^{-s}$ are bounded for all even integer s. By interpolation, they are bounded for all s. This proves the independence of the Sobolev space on the choice of a metric on M.

The last claim follows from Lemma 5 if s is an integer. Let $H^\infty := \bigcap H^k(\mathcal{H}, \varrho)$. Then $\varrho(P)(H^\infty) \subseteq H^\infty$. Using this fact and applying the Phragmen-Lindelöf principle to $s \mapsto (\varrho(D)^s P \varrho(D)^{-s}\xi, \xi')$, with $\xi, \xi' \in H^\infty$, we obtain the desired result for all s. $\square$

Similarly, we prove the following corollary.

Corollary 4. *Let $A \in \Psi^k(\mathcal{G})$ be elliptic. Then $\Lambda := \varrho(E + A^*A)$ is essentially self-adjoint, and Λ^t induces for all $s, t \in \mathbb{R}$ isomorphisms*

$$\Lambda^t : H^s(\mathcal{H}, \varrho) \longrightarrow H^{s-2kt}(\mathcal{H}, \varrho) .$$

Another corollary is related to the Cayley transform.

Corollary 5. *If $A = A^* \in \Psi^m(\mathcal{G})$, $m > 0$, is elliptic, then the Cayley transform $(A+iE)(A-iE)^{-1}$ of A belongs to $\mathfrak{A}(\mathcal{G})$. Moreover, we have*

$$\sigma_0((A+iE)(A-iE)^{-1}) = \sigma^m(A)\sigma_m(A)^{-1} = 1 ,$$

where the last equality holds in the scalar case only.

Proof. We have $(A+iE)(A-iE)^{-1} = (A+iE)^2(A^2+E)^{-1} \in \mathfrak{A}(\mathcal{G})$, by Theorem 4, because $A^2+E \geq E$ and $\sigma_{2m}(A^2+E) > 0$. The identity for the principal symbol follows from the corresponding one in Lemma 5. □

The Cayley transform of A will be denoted in the following sections simply by $(A+iE)(A-iE)^{-1}$, because no more confusions can arise.

8. Operators on open manifolds

One of the main motivations for studying algebras of pseudodifferential operators on groupoids is that they can be used to analyze geometric operators on certain complete Riemannian manifolds (M_0, g) (without corners). The groupoids $\mathcal{G}$ used to study these geometric operators will be of a particular kind. They will have as space of units a compactification M of M_0 to a manifold with corners such that M_0 will be an open invariant subset of M with the property that the reduction of $\mathcal{G}$ to M_0 is the product groupoid. If M_0 happens to be compact, then $M = M_0$, and our results simply reduce to the usual "elliptic package" for compact smooth manifolds without corners. Our results thus can be viewed as a generalization of the classical elliptic theory from compact manifolds to certain non-compact, complete Riemannian manifolds.

We now make explicit the hypothesis we need on the groupoid $\mathcal{G}$.

Assumptions. *In this and the following sections, M_0 will be a smooth manifold without corners which is diffeomorphic to (and will be identified with) an open dense subset of a compact manifold with corners M, and $\mathcal{G}$ will be a differentiable groupoid with units M, such that M_0 is an invariant subset and*

$$\mathcal{G}_{M_0} \cong M_0 \times M_0.$$

The above assumptions have a number of useful consequences for $\mathcal{G}$, M, and M_0, and we shall use them in what follows, without further comment.

Let $A = A(\mathcal{G})$. First of all, $A|_{M_0} \cong TM_0$. Fix a metric on A. The metric on A then restricts to a metric on M_0, so M_0 is naturally a Riemannian manifold such that the map $r : \mathcal{G}_x \to M_0$ is an isometry for any $x \in M_0$. Moreover, because M is compact, all metrics on M_0 obtained by this procedure will be equivalent: if g_1 and g_2 are metrics on M_0 obtained from metrics on A, then we can find $C, c > 0$ such that $cg_1 \leq g_2 \leq Cg_1$ (this is of course not true for any two metrics on the non-compact smooth manifold M_0). The same result holds true for the induced smooth densities (or measures) on M_0, and hence all the spaces $L^2(M_0)$ defined by these measures actually coincide.

Let π be the vector representation of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}^\infty(M)$ (uniquely determined by $(\pi(P)f)\circ r = P(f\circ r)$, see Equation (9)). Then $\pi(\Psi^\infty(\mathcal{G}))$ maps $\mathcal{C}_c^\infty(M_0)$ to itself. Fix $x \in M_0$. The regular representation $\pi_x : \Psi^\infty(\mathcal{G}) \to End(C_c^\infty(\mathcal{G}_x))$ is equivalent to π via the isometry $r : \mathcal{G}_x \to M_0$, and hence π is a bounded representation of $\Psi^\infty(\mathcal{G})$ on $L^2(M_0)$.

We now relate the geometric operators on M_0, defined using a metric induced from A, and the geometric operators on $\mathcal{G}$, $\mathcal{G}$ as above ($\mathcal{G}_{M_0} \cong M_0 \times M_0$).

We start with a $\mathrm{Cliff}(A)$-module W on M together with an admissible connection $\nabla^W \in \mathrm{Diff}(\mathcal{G}; W, W \otimes A^*)$, defined as an invariant family of differential operators on $\mathcal{G}_x = d^{-1}(x)$. Fix $x \in M_0$ arbitrary. Then the restriction of $r^*(W)$ to $\mathcal{G}_x$ is a Clifford module on $\mathcal{G}_x$, which hence can be identified with a Clifford module W_0 on M_0, using the isometry $\mathcal{G}_x \cong M_0$.

Let $D\!\!\!\!/^W \in \Psi^1(\mathcal{G}; W)$ be the Dirac operator on $\mathcal{G}$ associated to W and its admissible connection, and let $D\!\!\!\!/^{W_0}$ be the Dirac operator on M_0 associated to W_0 and its admissible connection obtained by pulling back the connection on $\mathcal{G}_x$. These operators are related as follows.

Theorem 5. *The Dirac operator $D\!\!\!\!/^W$ on $\mathcal{G}$ acts in the vector representation as $D\!\!\!\!/^{W_0}$, the Dirac operator on $M_0 \subset M$ defined above. More precisely,*

$$\pi(D\!\!\!\!/^W) = D\!\!\!\!/^{W_0}.$$

Proof. By construction, $D\!\!\!\!/^{W_0}$ is, up to similarity, the restriction of $D\!\!\!\!/^W$ to one of the fibers $\mathcal{G}_x$, with $x \in M_0$. □

At first sight, the above theorem applies only to a very limited class of (admissible) Dirac operators on M_0, the ones coming from $\mathrm{Cliff}(A)$-modules. Not every Dirac operator on a Clifford module on M_0 can be obtained in this way. However, as we shall see in a moment, if we are given a Clifford module on M_0, we can always adjust our compatible connection so that the resulting Dirac operator comes from a Dirac operator on $\mathcal{G}$ (corresponding to a $\mathrm{Cliff}(A)$-module).

Theorem 6. *Suppose there exists a compact subset $M_1 \subset M_0$ which is a deformation retract of M. Let W_0 be a Clifford module on M_0. Then we can find an admissible connection on W_0 such that the associated admissible Dirac operator $D\!\!\!\!/^{W_0}$ is (conjugate to) $\pi(D\!\!\!\!/^W)$, for some* $\mathrm{Cliff}(A)$*-module W, $D\!\!\!\!/^W$ being the Dirac operator on $\mathcal{G}$ associated to W.*

If W_0 is a spin bundle, then we can choose this connection to be the Levi-Civita connection on W_0.

Proof. Using the deformation retract $f : M \to M_1$, we define (up to isomorphism) $W = f^*(W_0)$. Then $W|_{M_0} \simeq W_0$, the isomorphism being uniquely determined up to homotopy. Moreover, we have a (non-canonical) isomorphism $A \simeq f^*(TM_0)$ of vector bundles, which allows us to define a $\mathrm{Cliff}(A)$-module structure on W. By replacing W_0 with an isomorphic bundle, we can assume then that $W_0 = W|_{M_0}$, as Clifford modules. Choose an admissible connection on W. Theorem 5 then gives that $\pi(D\!\!\!\!/^W) = D\!\!\!\!/^{W_0}$. □

9. Spectral properties

We shall use now the results of the previous section to study operators on suitable Riemannian manifolds. We are interested in spectral properties, Fredholmness,

and compactness for these operators. The results of this section extend essentially without change to the case of families of such manifolds.

We fix, throughout this section, a groupoid $\mathcal{G}$ satisfying the assumptions of Section 8. In particular, M_0 is an open invariant subset of M and $\mathcal{G}_{M_0} \cong M_0 \times M_0$.

We denote as before by $\mathfrak{A}(\mathcal{G})$ the closure of $\Psi^0(\mathcal{G})$ in the norm $\|\cdot\|$ and by $C^*(\mathcal{G})$ the closure of $\Psi^{-\infty}(\mathcal{G})$ in the same norm. Our analysis of geometric operators on M_0 depends on the structure of the algebras $\mathfrak{A}(\mathcal{G})$ and $C^*(\mathcal{G})$. The results of Section 4 applied to our groupoid $\mathcal{G}$ (satisfying the assumptions of Section 8) give the following. Let $\mathfrak{I} = C^*(\mathcal{G}_{M_0})$, then $\mathfrak{I}$ is isomorphic to $\mathcal{K}(L^2(M_0))$, the algebra of compact operators on $L^2(M_0) = L^2(M)$, the isomorphism being induced by the vector representation π, or by any of the representations π_x, $x \in M_0$, and the isometry $\mathcal{G}_x \simeq M$. Otherwise, if $x \notin M_0$, then π_x descends to a representation of $Q(\mathcal{G}) := \mathfrak{A}(\mathcal{G})/\mathfrak{I}$.

We shall study various spectra, for this purpose, the results of Section 7 will prove indispensable.

We denote by $\sigma(P)$ the spectrum of an element $P \in \mathfrak{A}(\mathcal{G})$ and by $\sigma_{Q(\mathcal{G})}(P)$ the spectrum of the image of P in $Q(\mathcal{G}) := \mathfrak{A}(\mathcal{G})/\mathfrak{I} = \mathfrak{A}(\mathcal{G})/C^*(\mathcal{G}_{M_0})$. These definitions extend to elliptic, self-adjoint elements $P \in \Psi^m(\mathcal{G})$, $m > 0$, using the Cayley transform, as follows. Let $f(t) = (t+i)/(t-i)$ and $f(P) := (P+i)(P-i)^{-1} \in \mathfrak{A}(\mathcal{G})$ be its Cayley transform, which is defined by Corollary 5. We define then

$$\sigma(P) := f^{-1}(\sigma(f(P)))\,, \quad \text{and} \quad \sigma_{Q(\mathcal{G})}(P) := f^{-1}(\sigma_{Q(\mathcal{G})}(f(P))). \tag{27}$$

We observe that if P is identified with an unbounded, self-adjoint operator on a Hilbert space, then the relation $\sigma(P) := f^{-1}(\sigma(f(P)))$ is automatically satisfied, by the spectral mapping theorem.

The spectrum and essential spectrum of an element T acting as an unbounded operator on a Hilbert space will be denoted by $\sigma(T)$ and, respectively, by $\sigma_{ess}(T)$. (We shall do that for an operator of the form $T = \pi(P)$, with π the vector representation and $P \in \Psi^m(\mathcal{G}; F)$, $m > 0$, elliptic, self-adjoint.)

We shall formulate all results below for operators acting on vector bundles. Fix an elliptic operator $A \in \Psi^m(\mathcal{G}; F)$, $m > 0$, then for any $P \in \Psi^k(\mathcal{G}; F)$, we have $P_1 := P(E + A^*A)^{-k/2m} \in \mathfrak{A}(\mathcal{G}; F)$, by Theorem 4.

Let us notice that if π is the vector representation of $\Psi^0(\mathcal{G})$ on $L^2(M) = L^2(M_0)$ (see Equation 9 for the definition of the vector representation), then the spaces $H^s(M) = H^s(L^2(M), \pi)$ are the usual Sobolev spaces associated to the Riemannian manifold of bounded geometry $\mathcal{G}_x \simeq M_0$ [40, 46]. If we are working with sections of a Hermitian vector bundle F, then we write $H^s(M; F) := H^s(L^2(M; F), \pi)$.

Theorem 7. *Let $M_0 \subset M$, the groupoid $\mathcal{G}$, and $A \in \Psi^k(\mathcal{G}; F)$, elliptic, be as above.*

(i) If $P \in \Psi^m(\mathcal{G}; F)$ is such that the image of $P_1 := P(E + A^*A)^{-m/2k} \in \mathfrak{A}(\mathcal{G})$ in $Q(\mathcal{G}) := \mathfrak{A}(\mathcal{G})/C^*(\mathcal{G}_{M_0})$ is invertible, then $\pi(P)$ extends to a Fredholm operator $H^m(M; F) \to L^2(M; F)$.

(ii) If $P_1 := P(E + A^*A)^{-m/2k}$ maps to zero in $Q(\mathcal{G})$, then $\pi(P)$ is a compact operator $H^m(M; F) \to L^2(M; F)$.

(iii) If $P \in \Psi^0(\mathcal{G}; F)$ or $P \in \Psi^m(\mathcal{G}; F)$, $m > 0$, is self-adjoint, elliptic, then $\sigma(\pi(P)) \subseteq \sigma(P)$ and $\sigma_{ess}(\pi(P)) \subseteq \sigma_{Q(\mathcal{G})}(P)$.

Proof. Let $P_1 := P(E + A^*A)^{-m/2k}$, as above.

(i) Choose $Q_1 \in \mathfrak{A}(\mathcal{G})$ such that

$$Q_1P_1 - E, P_1Q_1 - E \in \mathfrak{I} := C^*(\mathcal{G}_{M_0}),$$

and define $Q = \pi((E + A^*A)^{-m/2k}Q_1)$. Then $\pi(P)Q - \mathrm{id}_{L^2(M)} \in \pi(\mathfrak{I}) = \mathcal{K}$. Similarly, we find a right inverse for $\pi(P)$ up to compact operators. Thus, $\pi(P)$ is Fredholm.

(ii) The operator $\pi(P) : H^m(M; F) \to L^2(M; F)$ is the product of the bounded operator $\pi(E + A^*A)^{m/2k} : H^m(M; F) \to L^2(M; F)$ and of the compact operator $\pi(P_1)$.

(iii) For P in a C^*-algebra A_0 and ϱ a bounded $*$-representation of A_0, the spectrum of $\varrho(P)$ is contained in the spectrum of P (we do not exclude the case where they are equal). If $P \in \Psi^0(\mathcal{G}; F)$, this gives (iii), by taking $A_0 = \mathfrak{A}(\mathcal{G})$ or $A_0 = Q(\mathcal{G})$. If $P \in \Psi^m(\mathcal{G}; F)$, $m > 0$, is self-adjoint, elliptic, then we use the result we have just proved for $f(P) = (P + iE)(P - iE)^{-1}$, the Cayley transform of P. $\square$

It is interesting to observe the following. Both (i) and (ii) can be proved using (iii). However, because (i) and (ii) are more likely to be used, we also included separate, simpler proofs of (i) and (ii). A derivation of (i) and (ii) from (iii) can be obtained as in the proof of the following theorem.

Let $\pi : \Psi^\infty(\mathcal{G}) \longrightarrow \mathrm{End}(\mathcal{C}_c^\infty(M))$ be the vector representation. Then the homogeneous principal symbol $\sigma_0(P)$ of $P \in \Psi^0(\mathcal{G})$ can be recovered from the action of $\pi(P)$ on $\mathcal{C}^\infty(M)$ by oscillatory testing as in the classical case. Indeed, let $x \in M$ and $\xi \in A_x^*(\mathcal{G}) = T_x^*\mathcal{G}_x$ be arbitrary. Then we have

$$\sigma_0(P)(\xi) = \lim_{t \to \infty} \left[e^{-itf}\pi(P)\varphi e^{itf}\right](x) \tag{28}$$

for all $\varphi \in \mathcal{C}_c^\infty(M)$ with $\varphi = 1$ near x, and all $f \in \mathcal{C}^\infty(M, \mathbb{R})$ with $d(f \circ r) \neq 0$ on $\mathrm{supp}(\varphi \circ r)$, and $d(f \circ r)|_x = \xi$. The proof of (28) uses the fact that the homogeneous principal symbol $(\sigma_0(P)(\xi) = \sigma_0(P_x)(\xi))$ as well as the action of $\pi(P)$ on $\mathcal{C}_c^\infty(M)$ $((\pi(P)h)(x) = P_x(h \circ r)|_{\mathcal{G}_x}(x))$ are defined using the manifold $\mathcal{G}_x$ only. Thus, the classical result applies.

Suppose now that the vector representation $\pi : C^*(\mathcal{G}) \to \mathcal{B}(L^2(M))$ is injective. Then the above result can be sharpened to a necessary and sufficient condition for Fredholmness, respectively for compactness. We first note that since the principal symbol of a pseudodifferential operator can be determined from its action on functions, the representation $\pi : \mathfrak{A}(\mathcal{G}) \to \mathcal{B}(L^2(M))$ is also injective. Indeed, this follows from formula (28).

Theorem 8. *Assume that the vector representation π is injective on $C^*(\mathcal{G})$. Using the notation from the above theorem, we have.*

(i) If $P \in \Psi^m(\mathcal{G}; F)$ is such that $\pi(P)$ defines a Fredholm operator $H^m(M; F) \to L^2(M; F)$ then the image of $P(E + A^*A)^{-m/2k} \in \mathfrak{A}(\mathcal{G})$ in $Q(\mathcal{G}) := \mathfrak{A}(\mathcal{G})/C^*(\mathcal{G})$ is invertible.

(ii) If $\pi(P)$ defines a compact operator $H^m(M; F) \to L^2(M; F)$, then the image of $P(E + A^*A)^{-m/2k}$ in $Q(\mathcal{G})$ vanishes.

(iii) If $P \in \Psi^0(\mathcal{G}; F)$ or $P \in \Psi^m(\mathcal{G}; F)$, $m > 0$, is self-adjoint, elliptic, then $\sigma(\pi(P)) = \sigma(P)$ and $\sigma_{ess}(\pi(P)) = \sigma_{Q(\mathcal{G})}(P)$.

Proof. An injective representation π of C^*-algebras preserves the spectrum, and in particular, a is invertible if, and only if, $\pi(a)$ is invertible.

Denote by $\mathcal{B}$ the algebra of bounded operators on $L^2(M; F)$. The morphism $\pi' : Q(\mathcal{G}) \to \mathcal{B}/\mathcal{K}$ induced by π is also injective. Fix $P_0 \in \mathfrak{A}(\mathcal{G})$. Then P_0 is

invertible if, and only if, $\pi(P_0)$ is invertible. By replacing P_0 with $P_0 - \lambda E$, we obtain $\sigma(P_0) = \sigma(\pi(P_0))$. We see then that P_0 is invertible modulo $C^*(\mathcal{G}_{M_0})$ if, and only if, $\pi(P_0)$ is invertible modulo compact operators. This gives $\sigma_{Q(\mathcal{G})}(P_0) = \sigma_{ess}(\pi(P_0))$. We thus obtain (iii) if we take $P_0 = P$ or $P_0 = f(P)$, the Cayley transform of P.

(i) By definitions, $\pi(P_1)$ is Fredholm if, and only if, $\pi(P)$ defines a Fredholm operator $H^m(M;F) \to L^2(M;F)$. Then

$$\begin{aligned} \pi(P_1) \text{ is Fredholm} \quad &\Longleftrightarrow \quad 0 \notin \sigma_{ess}(\pi(P_1)) \\ &\Longleftrightarrow \quad 0 \notin \sigma_{Q(\mathcal{G})}(P_1) \\ &\Longleftrightarrow \quad P_1 \text{ is invertible in } Q(\mathcal{G})\,. \end{aligned}$$

For (ii), a similar reasoning holds:

$$\begin{aligned} \pi(P_1) \text{ is compact} \quad &\Longleftrightarrow \quad \sigma_{ess}(\pi(P_1^* P_1)) = \{0\} \\ &\Longleftrightarrow \quad \sigma_{Q(\mathcal{G})}(P_1^* P_1) = \{0\} \\ &\Longleftrightarrow \quad P_1 = 0 \in Q(\mathcal{G})\,. \end{aligned}$$

□

The criteria in the above theorems can be made even more explicit in particular examples.

Theorem 9. *Suppose the restriction of $\mathcal{G}$ to $M \smallsetminus M_0$ is amenable, and the vector π representation is injective. Then,*

(i) $P : H^s(M;F) \to L^2(M;F)$ is Fredholm if, and only if, P is elliptic and $\pi_x(P) : H^s(\mathcal{G}_x, r^*F) \to L^2(\mathcal{G}_x, r^*F)$ is invertible, for any $x \notin M_0$.

(ii) $P : H^s(M;F) \to L^2(M;F)$ is compact if, and only if, its principal symbol vanishes, and $\pi_x(P) = 0$, for all $x \notin M_0$.

(iii) For $P \in \Psi^0(\mathcal{G};F)$, we have

$$\sigma_{ess}(\pi(P)) = \bigcup_{x \notin M_0} \sigma(\pi_x(P)) \cup \bigcup_{\xi \in S^*\mathcal{G}} \operatorname{spec}(\sigma_0(P)(\xi))\,,$$

where $\operatorname{spec}(\sigma_0(P)(\xi))$ *denotes the spectrum of the linear map $\sigma_0(P)(\xi) : E_x \to E_x$.*

(iv) If $P \in \Psi^m(\mathcal{G};F)$, $m > 0$, is self-adjoint, elliptic, then we have $\sigma_{ess}(\pi(P)) = \cup_{x \notin M_0} \sigma(\pi_x(P))$.

Proof. Again, (i) and (ii) follow from (iii) and (iv). The assumption $\mathfrak{A}(\mathcal{G}) = \mathfrak{A}_r(\mathcal{G})$ implies $\mathfrak{A}(\mathcal{G})/\mathfrak{I} = \mathfrak{A}_r(\mathcal{G})/\mathfrak{I}$. Because the groupoid obtained by reducing $\mathcal{G}$ to $M \smallsetminus M_0$ is amenable, the representation $\varrho := \prod \pi_x$, $x \notin M_0$ is injective on $Q(\mathcal{G})$. This gives $\sigma_{Q(\mathcal{G})}(T) = \cup_x \sigma(\pi_x(T))$, $x \notin M_0$, for all $T \in \mathfrak{A}(\mathcal{G})$. □

Another explicit criterion is contained in the theorem below.

Theorem 10. *Suppose the vector representation π is injective and $M \setminus M_0$ can be written as a union $\bigcup_{j=1}^r Z_j$ of closed, invariant manifolds with corners $Z_j \subset M$.*

(i) Let $P \in \Psi^m(\mathcal{G};F)$, then $P : H^s(M) \to L^2(M)$ is Fredholm if, and only if, it is elliptic and $\mathcal{R}_{Z_j}(P) : H^s(Z_j) \to L^2(Z_j)$ is invertible, for all j.

(ii) Let $P \in \Psi^m(\mathcal{G};F)$, then $P : H^s(M) \to L^2(M)$ is compact if, and only if, its principal symbol vanishes and $\mathcal{R}_{Z_j}(P) = 0$, for all j.

(iii) For $P \in \Psi^0(\mathcal{G};F)$, we have

$$\sigma_{ess}(\pi(P)) = \bigcup_{j=1}^{r} \sigma(\mathcal{R}_{Z_j}(P)) \cup \bigcup_{\xi \in S^*\mathcal{G}} \operatorname{spec}(\sigma_0(P)(\xi))\,.$$

(iv) Suppose $P \in \Psi^m(\mathcal{G}; F)$, $m > 0$, is self-adjoint, elliptic. Then we have $\sigma_{ess}(\pi(P)) = \cup_{j=1}^r \sigma(\mathcal{R}_{Z_j}(P))$.

Proof. The representation $\mathfrak{A}(\mathcal{G})/\mathfrak{I} \to \oplus_j \mathfrak{A}(\mathcal{G}_{Z_j}) \oplus \mathcal{C}(S^*\mathcal{G}; \mathrm{End}(F))$ given by the restrictions $\mathcal{R}_{Z_j}$ and the homogeneous principal symbol is injective. This gives (iii) and (iv). For $m > 0$ note that we have $\sigma_0(f(P)) = \mathrm{id}_F$ for the Cayley transform $f(P) = (P + iE)(P - iE)^{-1} \in \mathfrak{A}(\mathcal{G})$ of P, and $f^{-1}(1) = \{\infty\}$.

To obtain (i) and (ii) from (iii) as above, it is enough to observe that the operator $P_1 = P(E + A^*A)^{-m/2k}$, (with A elliptic of order m, fixed) belongs to $\mathfrak{I} = C^*(\mathcal{G}_{M_0})$ if, and only if, $\mathcal{R}_{Z_j}(P_1) = 0$ for all j, and that $\mathcal{R}_{Z_j}(P_1) = 0$ if, and only if, $\mathcal{R}_{Z_j}(P) = 0$. □

In Section 10, we shall see examples of groupoids for which the conditions of the above theorem are satisfied. Having this natural characterization of Fredholmness, it is natural to ask for an index formula for these operators, at least in the case when the restriction of $\mathcal{G}$ to each component S of is such that $r(\mathcal{G}_x)$ has constant dimension for all x in a fixed component S of $Y_k \smallsetminus Y_{k-1}$ (that is, when the restriction of $A(\mathcal{G})$ to each component S of $Y_k \smallsetminus Y_{k-1}$ is a regular Lie algebroid). The results of [30] deal with a particular case of this problem, when $M = Y_n$, the induced foliation on M is a fiber bundle, and the isotropy bundle can be integrated to a bundle of Lie groups that consists either of compact, connected Lie groups or of simply-connected, solvable Lie groups.

10. Examples III: Applications

For geometric operators P, the operators $\pi_x(P)$ and $\pi_Z(P)$ appearing in the statements of the above theorems are again geometric operators of the same kind (Dirac, Laplace, ...). This leads to very explicit criteria for their Fredholmness and to the inductive determination of their spectrum.

Example 23. If $\mathcal{G} = M \times M$ is the pair groupoid, then $C^*(\mathcal{G}) \cong \mathcal{K} = \mathcal{K}(L^2(M))$ and all the results stated above were known for these algebras. In particular, the exact sequence

$$0 \to \mathcal{K} \to \mathfrak{A}(\mathcal{G}) \to \mathcal{C}(S^*M) \to 0$$

is well-known. Moreover, the criteria for compactness and Fredholmness are part of the classical elliptic theory on compact manifolds. There is no need for an inductive determination of the spectrum in this case.

We are now going to apply the results of the previous section to the c_n-calculus considered in Example 21. The main result being an inductive method for the determination of the essential spectrum of Hodge-Laplace operators. Because the b-calculus corresponds to the special case $c_H = 1$ for all boundary hyperfaces H of M, we in particular answer an question of Melrose on the essential spectrum of the b-Laplacian on a compact manifold M with corners [21, Conjecture 7.1].

Let $\mathcal{G}(M, c)$ be the groupoid constructed in Example 21 for an arbitrary system $c = (c_H)$.

Lemma 7. *The groupoid $\mathcal{G}(M, c)$ is amenable and the vector representation of $\mathfrak{A}(\mathcal{G}(M, c))$ is injective.*

Proof. It is enough to prove that the representation π is injective on $C^*(\mathcal{G}(M, c))$, because we can recover the principal symbol of a pseudodifferential operator from its action on functions, as explained in the previous section.

The groupoid $\mathcal{G}$ is amenable because the composition series of Theorem 3 are associated to the groupoids $(S \times S) \times \mathbb{R}^k$, which are amenable groupoids.

It is then enough to prove that each representation of the form π_x is contained in the vector representation. Let $x \in F$ be an interior point. By considering a small open subset of x, we can reduce the problem to the case when the manifold M is of the form $[0, \infty)^k \times \mathbb{R}^{n-k}$. Then the result is reduced to the case $k = n = 1$ using Proposition 2. But for this case $C^*(\mathcal{G})$ is isomorphic to the crossed product algebra $C_0(\mathbb{R} \cup \{-\infty\}) \rtimes \mathbb{R}$ and the vector representation corresponds to the natural representation on $L^2(\mathbb{R})$ in which $C_0(\mathbb{R} \cup \{-\infty\})$ acts by multiplication and $\mathbb{R}$ acts by translation. This representation is injective (it is actually often used to define this crossed product algebra). From this the result follows. $\square$

The algebra $C_0(\mathbb{R} \cup \{-\infty\}) \rtimes \mathbb{R}$ is usually called the algebra of *Wiener-Hopf operators* on $\mathbb{R}$, for which it is well-known that the vector representation is injective.

Fix now a metric h on $A = A(\mathcal{G}(M, c))$, and let $\Delta_p^c := \Delta_p^{\mathcal{G}(M,c)}$ be the corresponding Hodge-Laplacian acting on p-forms. Note that each boundary hyperface H of M is a closed, invariant submanifold with corners, whereas the interior $M_0 := M \setminus \partial M = M \setminus \bigcup H$ is invariant and satisfies $\mathcal{G}_{M_0} \cong M_0 \times M_0$. We are in position then to use the results of the previous section.

First we need some notation. For each hyperface H of M, we consider the system $c^{(H)}$ determined by $c_F^{(H)} = c_{F'}$ for all boundary hyperfaces F' of M with $F := H \cap F' \neq \emptyset$, as in the Example 21. By the construction of the groupoid $\mathcal{G}(M, c)$,

$$\mathcal{G}(M, c)_H \cong \mathcal{G}(H, c^{(H)}) \times \mathbb{R}.$$

It will be convenient to use the Fourier transform to switch to the dual representation in the $\mathbb{R}$ variable, so that the action of the group by translation becomes an action by multiplication. Then pseudodifferential operators on $\mathcal{G}(M, c)_H$ become families of pseudodifferential operators on $\mathcal{G}(H, c^{(H)})$ parametrized by $\mathbb{R}$. Using also (23), this reasoning then gives

$$\mathcal{R}_H(\Delta_p^c) = \Delta_p^{\mathcal{G}(M,c)_H} = \begin{cases} \lambda^2 + \Delta_0^{c^{(H)}}, & \text{if } p = 0, \\ \big(\lambda^2 + \Delta_p^{c^{(H)}}\big) \oplus \big(\lambda^2 + \Delta_{p-1}^{c^{(H)}}\big), & \text{if } p > 0, \end{cases} \tag{29}$$

because for $p > 0$ the space of p-forms on the product with $[0, \infty)$ splits into the product of the spaces of $p-1$ and p forms that contain, respectively, do not contain, dt, $t \in [0, \infty)$.

Denote by $m_H^{(p)} = \min \sigma(\Delta_p^{c^{(H)}})$ and by $m^{(p)} = \min_H m_H^{(p)}$. Then $m^{(p)} \geq 0$ because the Hodge-Laplace operators $\Delta_p^{c^{(H)}}$ are positive operators.

On the other hand, note that $\pi(\Delta_p^c)$ is (conjugated to) Δ_p, the Hodge-Laplace operator acting on p-forms on the complete manifold $M_0 := M \smallsetminus \partial M$, with the induced metric from $A(M, c)$.

Theorem 11. *Consider the manifold M_0, which is the interior of a compact manifold with corners M, with the metric induced from $A(M, c)$. Then the essential spectrum of the (closure of the) Hodge-Laplacian Δ_p acting on p-forms on M_0 is $[m, \infty)$, with $m = m^{(0)}$, if $p = 0$, or $m = \min\{m^{(p)}, m^{(p-1)}\}$, if $p > 0$, using the notation explained above.*

In particular, the spectrum of Δ_p itself is the union of $[m, \infty)$ and a discrete set consisting of eigenvalues of finite multiplicity.

Proof. We are going to apply Theorem 10 (iv), with Z_j ranging through the set of hyperfaces of M; this is possible because of Lemma 7. Furthermore, note that by the definition of the groupoid structure on $\mathcal{G}(M,c)$ in Example 21, the boundary hyperfaces H of M are closed, invariant submanifolds with $M \setminus M_0 = \bigcup_H H$.

For each boundary hyperface H of M, we have by (29)

$$\sigma(\mathcal{R}_H(\Delta_p^c)) = \bigcup_{\lambda\in\mathbb{R}} \left(\lambda^2 + \left(\sigma(\Delta_p^{c^{(H)}}) \cup \sigma(\Delta_{p-1}^{c^{(H)}})\right)\right) = [\min\{m_H^{(p-1)}, m_H^{(p)}\}, \infty)\,,$$

where $\lambda^2 + \sigma(\Delta_{p-1}^{c^{(H)}})$ is missing if $p = 0$. Since Δ_p is essentially self-adjoint and elliptic, Theorem 10 (iv) completes the proof. □

Using an obvious inductive procedure, we then obtain the following more precise result on the spectrum of the Laplace operator acting on functions. [1]

Corollary 6. *Let M_0 be as above, then the spectrum of the (closure of the) Laplace operator Δ_0 on M_0 is $\sigma(\Delta_0) = [0,\infty)$, and hence it coincides with its essential spectrum.*

Proof. Let F be a minimal face of M (that is, not containing any other face of M). Then F is a compact manifold without corners and hence the Laplace operator on F contains 0 in its spectrum. The above theorem then shows that $[0,\infty) \subset \sigma_{ess}(\Delta_0)$. On the other hand, Δ_0 is positive, and hence $\sigma(\Delta_0) \subset [0,\infty)$. This completes the proof. □

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